Children's Expressions of Randomness: Constructing Probabilistic Ideas In an Open Computer Game

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ABSTRACT

The research literature on children's understanding of randomness has developed considerably in recent decades, notably due to the key contributions of researchers such as Piaget and Inhelder (1975) and Tversky and Kahneman (1983). Yet the research has paid rather scant attention to the tools that people have available for expressing ideas about randomness, fairness and more generally, probability. In contrast, work within the paradigm of 'constructionism' makes the explicit claim that by using tools that are specially designed for expressing concepts of randomness and chance, people may be better able to express ideas that can seldom be predicted by cognitive analysis based on, say, misconceptions or thinking stages that fail to take sufficient account of tool mediation.

This study investigated the nature of young children's expressions of random events. Specifically the aims of the study were:

- iteratively to design and evaluate a tool-based game to afford children between the ages of 5 ½ and 8 opportunities to express and develop probabilistic ideas; and
- to describe and analyse how the tool-based game mediated the children's expressions of chance events.

An open computer game was designed for children to express understandings of randomness as formal conjectures, so that they were able to examine the consequences of their understandings. The game was designed simultaneously to afford children the opportunity to explore and express their intuitions and ideas, and to give the researcher the opportunity to study how probabilistic ideas evolved during the activity.

The study was organised in two main phases. The first, *iterative design* phase, compared two cycles of design and experiments with children as they played with and reconstructed the game. The second phase consisted of the *learning investigation* phase, which describes in detail the expressed ideas of children in using the game. This thesis shows how a visible and 'continuous' medium, i.e. one in which the sample space is represented by a spatial and dynamic metaphor, can enhance young children's expressions of randomness. The findings identify children's initial meanings for expressing stochastic phenomena and describe how the computer tool-based game helped to shift children's attempts to understand randomness from looking for ways to control random behaviour, towards looking for ways to control events. This was significant, since the study analyses how

children constructed their own ideas for fairness and in particular, how they constructed both symmetrical and asymmetrical spatial arrangements for it. In general, it is conjectured that the structure of the game, and in particular, the linkage between its components, assisted children in developing associated mental structures that developed their understandings of chance. Finally, evidence is presented that the children constructed a set of 'situated abstractions' for ideas such as 'distribution' and the 'law of large numbers'.

Στους γονείς μου,
Γιώτα και Χριστόφορο

To my parents,
Yiota and Christoforos

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CHAPTER ONE

The Research Rationale

1.1 Overview

This thesis is concerned with studying the way children can express ideas of randomness and probability with special focus on the tools they have available to express themselves. This chapter starts by giving the reasons for my decision to choose probability as the subject of this study. I use the term 'probability', to denote the branch of mathematics that describes randomness, and as a label with which I associate concepts like event, sample space and distribution. I outline here the motivations that made me interested in the subject and in so doing some aspects of my investigation come to the surface. The concept of probability, its history and my focus on subjective and intuitive probability from an epistemological view are then described in order to give a more complete idea of the subject. Finally, the overview of the thesis is presented.

1.2 Why probability?

At least two motivating factors have come into play in guiding my decision to select probability within the field of mathematics education as the focus of my research.

The first factor is that my personal interest contrasts with many people's distaste of the subject. I have always been fascinated by the fact that concepts and intuitions about probability theory are present in many aspects of our 'non-mathematical' lives. Intuitively, people make probabilistic judgements when they choose which is the safest way to cross a road, when they decide to park their car in the first available spot or take the risk and move on to find a place closer to their destination; when they change their diet to lower the risk of heart disease; when they select a school with the highest results in national examinations. Intuitive probability is something that is widely applied in everyday life, in making decisions and understanding social and natural phenomena.

Despite all this potential power, probability, from my personal experience as a student and as a teacher, is 'the subject everyone loves to hate' (Wilensky, 1995). Moreover, Garfield and Ahlgren (1988) state that many students develop a distaste of probability through having been exposed to its study in a highly abstract and formal way. A particular motivation for me to understand how children could better learn about probability is the widespread negative view of probability.

The second motivating factor is the existing work on probability in the field of mathematics education, which intrigued me to research the subject more extensively. Significant work in this field is Piaget and Inhelder's (1975) experiments on probabilistic judgements and the cognitive stages that they proposed based on these experiments. I will borrow here Piaget and Inhelder's own words that gradually defined my own stance towards the subject:

> ' Could there be in a normal man an intuition of probability just as fundamental and just as frequently used as, say, the intuition of whole numbers? Almost every common action seems, in fact, to require the notion of chance as well as a sort of spontaneous estimate of the more or less probable character of feared or expected events...Is such an intuition in-born or does it develop later and, if so, how is it acquired?' (p. xiii-xiv).

I found the above question interesting. Are probabilistic judgements made, developed or learned? And if they are developed, how does this happen?

A preliminary search of the literature seemed to reveal that there is a gap between intuitions and the construction of the formal concept of probability. In particular there is a gap between early intuitions about probability from everyday life and the formal knowledge gained from teaching.

Although there are many studies on conceptions of probability, most of them focus on students' 'misconceptions' (e.g. Tversky and Kahneman, 1983). An example of what Tversky and Kahneman (1983) mean by a misconception is the following: they undertook an experiment in which they asked people whether there are more words in the English language that begin with 'r' or that have 'r' as their third letter. Most people, wrongly, said that there are more words that begin with 'r'. Tversky and Kahneman argued that this error is attributable to a heuristic of 'availability' because people can much more easily recall words that begin with 'r' than words with 'r' in the third position. Since words beginning with 'r' are more available to them, they perceive them as more likely. This work describes an interesting view of how people make decisions without considering any probabilistic thinking, but what might it give to the *learning* of probability?

My reading of the literature led to a focus on the concept of probability at an early age, children aged 5½-8, where formal probabilistic knowledge is still undeveloped and intuitive knowledge plays a major role. I further decided to focus on the ideas of *sample space* and *distribution*. In the following sections I will describe some aspects of the history and epistemology of probability and I will outline the central importance of sample space, distribution and probability of an event.

1.3 The concept of probability

The theory of probability, or 'stochastics' as it is more commonly known in continental Europe, is the area of mathematics concerned with an attitude of doubt, or degree of belief, with respect to the outcome of some future event (Ferguson, 1971). In this section I will define what I mean by the terms 'randomness', 'probability', 'random phenomenon', 'sample space', 'event', 'probability of an event' and 'distribution'.

1.3.1 Randomness and probability

It is natural for humans to use statements of probability to describe uncertainty of the external world. People speak in everyday terms of 'chance' and 'randomness'. These concepts often serve them well in everyday communication because of the general consensus about their meanings. Yet, randomness is one of the most elusive concepts in mathematics. Hacking (1975) suggested that the meaning of the word random could be answered briefly, but it would take 100 pages to prove any answer correct!

This study adopts the definition of randomness and probability offered by Moore (1990):

> 'Phenomena having uncertain individual outcomes but a regular pattern of outcomes in many repetitions are called *random*. 'Random' is not a synonym for 'haphazard', but a description of a kind of order different from the deterministic one that is popularly associated with science and mathematics. Probability is the branch of mathematics that describes randomness.' (p.98)

Moore (1990) implies an external source of the uncertainty, referring to the external world, and opposed to an internal source in the form of one's knowledge. Falk, Falk, and Levin (1980) argue that probability is composed of two subconcepts: chance and proportion. One has to be aware of the uncertain nature of a situation in order to apply the results of proportional computations. Obviously the ability to calculate proportions as such does not necessarily signify understanding of probability. A realisation of uncertainty either in controlling or in predicting the outcome of an event is crucial.

Kuzmak and Gelman (1986) define the term 'random phenomenon' as follows:

> 'By a "random phenomenon," we mean a physical phenomenon that is conventionally viewed as having a number of equally probable outcomes (e.g., the roll of a die or the toss of a coin).' (p. 559)

They add that there are two basic characteristics of random phenomena: (a) details of the mechanism by which outcomes are produced are uncertain, from which it follows that (b) the individual outcomes of the phenomenon are unpredictable.

Heyman and Henriksen (1998) argue that probability can arise from two sources: the randomness of events in the world, and ignorance. Based on this view, someone who chooses heads or tails in a coin toss has 0.5 probability of being correct because of the inherent randomness of coin tossing. On the other hand, a person who loses his/her way and chooses to turn left or right has a 0.5 probability of going in the right direction given a lack of knowledge of the geography of the area. In other words, uncertainty about the future, from this perspective, may arise out of either the randomness of the world or lack of knowledge. The distinction between chance and lack of knowledge turns not only on the complexity, relative to observers' meanings, of the future that is being predicted, but also on their pragmatic concerns.

Descriptions of the chance of something happening are generally intended to downplay ignorance, while accounts of lack of knowledge point attention to the need to learn more. A quantitative description of the chance of an event occurring asserts implicitly that we have learnt all we can about it. This form of description invites us to stop asking questions and to start acting.

Chance is linked with other terms like 'risk', 'accident', 'opportunity' or 'possibility'. Probability can be thought of as the mathematical approach to the quantification of chance, just as rulers measure distances. The idea of risk reflects a fundamental yearning in humans, and perhaps all self-conscious beings, to know what will be; risk is a cultural concept employed in modern negotiations of everyday life as when, for example, somebody fastens his/her seatbelt before driving a car. The mathematical theory of risk provides a methodology for attempting to predict and control the future. It can also be defined as the projection of a degree of uncertainty about the future on the external world. The Royal Society (according to Heyman, 1998) defined risk as the probability that a particular adverse event occurs during a stated period of time, or resulting from a particular challenge.

Next, I will give a more detailed definition of sample space, event, probability of an event and distribution and these will be the probabilistic concepts that I focus on for my study. In particular, I will use sample space and distribution as 'windows' (Noss and Hoyles, 1996) into children's meaning making about randomness, probability of an event and probability comparisons.

1.3.2 Sample space, event, probability of an event, distribution

Consider an experiment whose outcome is not predictable with certainty. Although the outcome of the experiment will not be known in advance, let us suppose that the set of all possible outcomes is known. This set of all possible outcomes of an experiment is known as the *sample space* of the experiment and is denoted by S. For example, if the outcome of an experiment consists in the determination of the sex of a newborn child, then $S = \{g, b\}$, where the outcome g means that the child is a girl and b that it is a boy.

Any subset E of the sample space is known as an *event*. That is, an event is a set consisting of possible outcomes of the experiment. If the outcome of the experiment is contained in E, then we say that E has occurred. For example, if $E=\{g\}$, then E is the event that the child is a girl. Similarly if $F=\{b\}$, then F is the event that the child is a boy. The intersection of two events $(A \cap B)$is the event that can be described by saying that 'both A and B occur'. The union of two events $(A \cup B)$ is the event that 'either A or B occurs'. We shall distinguish between compound (or decomposable) and simple events. For example, consider the age of a person. Every particular value x represents a single event, whereas the statement that a person is in his fifties describes the compound event that x lies between 50 and 60. In this way every compound event can be decomposed into simple events, that is to say, a compound event is an aggregate of certain simple events. Following the general usage in mathematics, simple events will be called sample points, or points for short. By definition, every indecomposable result of the (idealized) experiment is represented by one, and only one, sample point. The aggregate of all sample points will be called the sample space. All events connected with a given (idealized) experiment can be described as aggregates of sample points.

In general, as Acredolo, O'Connor, Banks and Horobin (1989) define, the probability of an event is expressed as a ratio of the number of potential outcomes that may be considered successful over the number of all possible outcomes, successful plus unsuccessful. Grimment and Stirzaker (1992) defined it as follows:

> 'Suppose that we repeat an experiment a large number N of times, keeping the initial conditions as equal as possible, and suppose that A is some event which may or may not occur on each repetition. Our experience of most scientific experimentation is that the proportion of times that A occurs settles down to some value as N becomes larger and larger; that is to say, writing N (A) for the number of occurrences of A in the N trials, N (A)/N converges to a constant limit as N increases. We can think of the ultimate value of this ratio as being the probability **P** (A) that A occurs on any particular trial…'(p.4-5).

One way of defining the *probability of an event* is in terms of its relative frequency. Such a definition usually goes as follows: consider that an experiment, whose sample space is *S*, we define *n(E)* to be the number of times in the first n repetitions of the experiment that the event *E* occurs. Then *P(E)*, the probability of the event *E*, is defined by $P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$.

That is, *P(E)* is defined as the (limiting) proportion of the time that *E* occurs. We shall assume that for each event *E* in the sample space *S* there exists a value *P(E)*, referred to as the *probability of E.* We shall then assume that the probabilities satisfy a certain set of axioms. Consider an experiment whose sample space is *S*. For each event *E* of the sample space *S* we assume that a number *P(E)* is defined and satisfies the following three axioms.

Axiom 1: $0 \le P(E) \le 1$, the probability that the outcome of the experiment is an outcome in *E* is some number between 0 and 1,

Axiom 2: *P(S)*=1, with probability 1, the outcome must be a point in the sample space *S*, Axiom 3: $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$, for any sequence of mutually exclusive events the probability of at least one of these events occurring is just the sum of their respective probabilities.

It is often required to find the probability of an event *B* under the condition that an event *A* occurs. This probability is called the *conditional probability* of B given A and is denoted by *P(B/A)*. In this case A serves as a new (reduced) sample space, and that the probability is the fraction of *P(A)*, which corresponds to $A \cap B$. Thus *P(A/B)* = P($A \cap B$)/P(A), when *A* and *B* are events in a sample space and *A* has a nonzero probability.

The *distribution* of a random variable is concerned with the way in which the probability of its taking a certain value, or a value within a certain interval, varies. More commonly, the distribution of a discrete random variable is given by its 'probability mass function' and that of a continuous random variable by its 'probability density function'.

I will also illustrate here the *strong law of large numbers*, as this is probably the best-known result in probability theory. It states that the average of a sequence of independent random variables having a common distribution will, with probability 1, converge to the mean of that distribution. The theorem of the strong law of large numbers says:
Let X_1, X_2, ... be a sequence of independent and identically distributed random variables, each having a finite mean $\mu = E\,[X_i]$. Then, with probability 1,

$$\frac{X_1 + X_2 + \ldots + X_n}{n} \to \mu \quad \text{as} \quad n \to \infty .$$

(Feller, 1968; Clapham, 1990; Ross, 2002)

To conclude, probability theory supplies information about the likelihood that data could have resulted from chance alone and it can be used to make decisions about uncertain outcomes. A basic understanding of sample space is exhibited by the ability to identify the complete set of outcomes in a one-stage experiment (e.g. by throwing one die). Borovcnik and Bentz (1991) suggest that symmetry played a key role in the history of probability. They claim that Laplace's attempt to define probability is characterized by an intimate intermixture of sample space (the mathematical part) and the idea of symmetry (the intuitive part). Although, in modern mathematics, the sample space is completely separated from the probability, which is a function, defined on a specific class of subsets of sample space, the concept of sample space cannot fully be understood if it is not related to intuitions of symmetry.

Piaget and Inhelder (1975) suggested that children before eight years of age are 'prelogical', but after eight years of age are able to identify all possible outcomes in a one-stage experiment. However, Jones, Langrall, Thornton and Mogill (1997) reported that significant numbers of children in elementary school (before 11 years of age) were not able to list the outcomes of a one-stage experiment. From the sample space one can identify whether an event is possible, and its possibility, impossibility or certainty (Jones et al, 1997). (The meanings of these terms are as follows: 'Possible': $0<P(E)<1$; 'Impossible': $P(E)=0$; 'Certain': $P(E)=1$).

Understanding of the *probability of an event*, for the purposes of this study, is exhibited by the ability to identify and justify which of two or three events are most likely or least likely to occur (e.g. to understand the behaviour of an unfair die, which has one of six numbers twice). Many researchers have investigated young children's understanding of probability of an event (e.g. Acredolo et al, 1989; Piaget and Inhelder, 1975). Piaget and Inhelder (1975) concluded that young children base their probability predictions on a number of criteria, including both subjective and quantitative reasoning. According to their theory, young children have difficulty comprehending part-whole relationships. Acredolo, et al (1989) note that children commit themselves to one of three strategies in comparing event probabilities:

(a) a numerator strategy in which they only examine the part that corresponds to the event
(b) an incomplete denominator strategy in which they examine the part that corresponds to the complement of the event, and
(c) an integrating strategy in which they recognise the moderating effect that each part has on the other.

The next section will give a brief history of the concept of probability. The following paragraphs will try to give a 'global' idea of the 'genesis' of the concept of probability, connected in a way with people's intuitions about chance events.

1.4 Some selected aspects of the history of probability

Although we think of probabilities and risks as natural, taken-for-granted phenomena, history shows that our modern machinery for thinking about probability was invented relatively recently. A historical perspective provides a valuable corrective to the view that probability refers to a natural property of the world, rather than to one's understanding of it. Traces of the uses of probability in games of chance can be found in the ancient cultures of Indians, Babylonians, and Egyptians. The earliest known object for games of chance (around 3500 B.C.) is the astragalus, made from a bone in the heel of a sheep. Betting games with these bones were popular among Roman soldiers. It is possible that rubbing the round sides of the astragalus until they were approximately flat would make a primitive die. The cubes made from well-fired pottery, which were in use in Babylon 3000 B.C., were nearly perfect dice. Considerable experience would have been gained from casting dice or drawing beans out of urns for divine judgement at religious ceremonies (for example at Delphi in Greece) and it is curious that the conceptual breakthrough based on the regularity of the fall of dice did not occur before the birth of Christ.

The meaning of the term 'probable' has shifted qualitatively between medieval and modern times and the development of the modern concept of probability depended on a number of related shifts in the ways in which the world was understood (David, 1962; Hacking, 1975; Heyman and Henriksen, 1998). Formal theorising about probability emerged quite suddenly in the Western world in the second half of the seventeenth century, even though evidence of interest in odds and gambling had been found frequently in the Ancient world. There is disagreement about why formal theorising did not occur earlier and why it suddenly appeared at this time. Whatever the reasons, concern with the nature of probability and risk emerged as a defining feature of the modern Western world, bound up with the associated developments of science, trade and capitalism (Hacking, 1975). As Heyman and Henriksen (1998) state, some historical imagination is needed to visualise a worldview that did not take for granted our modern axioms about probability; for example, the 'law of averages' was unknown because the mathematical average had not been invented.

The field of probability and statistics is barely a mathematical adolescent when compared to geometry or to algebra, and even to the roots of the calculus are traceable back to Eudoxes and Archimedes. Hacking (1975) reviews a number of possible reasons why probability has been a late bloomer. He claims that this slow emergence was primarily due to the dual meaning that has historically been attached to the word probability and to the respective definitions of scientific evidence that accompanied each of these two meanings. David (1962) gives to Cardano in 1560 as the first reference to probability. Cardano discusses that there is equal probability of obtaining one half of the total number on the faces of a die as getting the other half. Thus, a die is honest if the wagers therefore are laid in accordance with this equality.

The seventeenth century saw the first steps taken towards an explicit theory of probability. Pascal and Fermat made great progress in conceptualising probability in their famous correspondence that was published in 1679. They solved the Division of Stakes (problem of points). Pascal and Fermat's achievement in the problem of points was to be the first to model the fair division of stakes by a game of chance. The problem deals with the fair division of stakes if a series of games has to be stopped before completion. At the beginning of the series two players bet equal stakes. The player who wins a certain number of single games first wins the whole stake. The series has to be interrupted before one of the players has reached the required number of points and the stakes have to be divided. If five games are required to win, and the score is 4:3 in favour of A, what is the fair division of stakes? The theory analysed introduced what should happen if the game was continued and if the chances of the players were equal for a single round. The stakes should be divided proportionally to the probability of winning in this continuation of the games. Pascal developed his famous arithmetic triangle as a general method to solve similar problems. The ideas of Fermat and Pascal were taken up by Bernoulli and were soon refined and developed for a vast variety of scientific purposes. Bernoulli used a maximum likelihood argument to prove that the best choice from a series of observations can differ from the arithmetic mean by using a particular continuous distribution of error. Bernoulli's *theorema aureum* in 1713 refers to the law of large numbers that relates individual probability to the probabilistic 'convergence' of relative frequencies. The stabilising of frequencies is a very intuitive means of transferring abstract probability onto the frequencies in large series. Francis Bacon, also in seventeenth century, stated the systematic and sufficiently precise development of Baconian probability and he gave other seminal ideas about non-demonstrative inference. Another well-known statistician of the seventeenth century was Abraham de Moivre. De Moivre was the first to find the function, which is now called the normal density.

In the 18th century Laplace made one of the most important statistical achievements by deriving the central limit theorem that states the binomial distribution approaches the normal distribution as the number of trials increases to infinity. Laplace came to the central limit theorem from his observations that errors of measurement, which can usually be regarded as being the sum of a large number of tiny forces, tend to be normally distributed. The application of the central limit theorem to show that measurement errors are approximately normally distributed is regarded as an important contribution to scientific method. Gauss used the normal distribution not only as a tool for approximation, but also as a distribution in its own right. He explored the relationship between four concepts; the mean as the best value to take from a series of measurements; the normal distribution for describing variation of errors; the maximum likelihood method to take the best value from a series of measurements; and the method of least squares to derive the best value replacing a series of measurements. Further details on history of probability are given in David (1962), Hacking (1975), Cohen (1979), Shaughnessy (1992), Daly, Hand, Jones, Lunn and McConway (1995), Borovcnik and Peard (1996), Ross (2002).

As it has been already mentioned, the historical perspective shows that the 'genesis' of the concept of probability is connected with people's intuitions. The next section describes the different categories of concept and it focuses on the 'subjective' and 'intuitive' probability.

1.5 A focus on 'subjective and intuitive probability'

The literature on probability (Hawkins and Kapadia, 1984; Konold, 1989; Shaughnessy, 1992; Wilensky, 1993) discusses several different categories of probability. These categories are distinguished as classical, frequentist, formal and subjective. The first three categories are the classical ones of teaching and learning probability. The thesis is focused on subjective and intuitive probability, as it applies to the subjects of this research.

Classical probability refers to the assignment of probabilities in an experiment with a random device where all outcomes are equally likely. Classical probability could be called the uniform probability distribution (Shaughnessy, 1992).

Frequentist probability considers probabilities that are limiting ratios of frequencies. When tossing a coin many times, we record the ratio of numbers of heads to number tosses. As the number of tosses increases without bound, this ratio reaches the probability of throwing a head. Mathematically, this involves the theory of limits and convergence (Konold, 1989; Wilensky, 1993).

Formal probability is the probability that is calculated precisely using the mathematical laws of probability. This is sometimes known as 'objective' or 'normative' probability. Formal probability requires some acquaintance with fractions while subjective probability may rely merely on comparisons of perceived likelihood.

Subjective and intuitive probability is the 20^{th} century term for opinion or degree of belief. When tossing a coin, the probability of being the outcome heads is relative to the beliefs of the coin tosser. The distinction between subjective and intuitive probabilities poses the greatest difficulty and as a result attracts the greatest amount of controversy and ambiguity. One view that has been put forward is that probability intuitions constitute notions of what is the 'correct' solution to a probabilistic problem, 'correct' being that which would accord with formal or theoretical probability. More details about what is intuition will be given in Chapter Two (section 2.2.1), as intuitions play an important role in this study. Subjective probability judgements on the other hand are said to be concerned with weighing evidence when there is no formal approach. Subjective and intuitive probability could therefore logically be accessible to less mathematically sophisticated children, at earlier stages of their mathematical education, than formal probability (Hawkins and Kapadia, 1984). Subjective and intuitive probability is an area that is often neglected in classroom oriented research, although it may be a fundamental precursor for the formal probability taught in schools (Hawkins and Kapadia, 1984; Wilensky, 1993). Hawkins and Kapadia (1984) conclude that a better understanding of growth and communication of probabilistic notions will not be achieved unless we include a consideration of the nature and influence of subjective and intuitive probabilities in the development of formal probability concepts.

1.6 Overview of the thesis

The aims of this study were concerned with designing a tool-based game to be used for children to express understandings of randomness as formal conjectures, so that they were able to examine the consequences of their understandings. The game was designed simultaneously to afford children the opportunity to explore and express their intuitions and ideas, and to give the researcher the opportunity to study how the game mediated the children's expressions of chance events.

The thesis is organised in nine chapters. Chapter One has provided the rationale of the study. The next chapter, Chapter Two, reviews the literature concerning the research in understanding the learning of the concepts of probability and indicates some didactic implications. It describes what intuitions are, and examines the idea of conceptions and misconceptions of probability. Furthermore, Chapter Two describes 'constructionist approaches', the idea of 'situated abstraction' and computer-based microworlds for stochastics. Chapter Three describes the aims of the study. It defines the aims of the research and provides an overview of the structure and the design principles of the tool-based game. The study is organised in two main phases: the iterative design phase and the learning investigation phase. The methodological issues of the two phases that comprised the study are presented in Chapter Four. Chapter Five begins the data analysis, focusing on the iterations of iterative design phase of the research. It also gives a description of the evolution of the game used for the learning investigation phase. Chapters Six to Eight present the main data analysis. Specifically, Chapter Six analyses children's thinking about sample space and global outcomes; Chapter Seven describes children's construction of the idea of fairness; Chapter Eight describes children's quantitative ideas of randomness, such as equality, the law of large numbers, possibility and proportional thinking. Finally, Chapter Nine provides the conclusions of the study, outlining some didactical implications, limitations of the study and suggestions for further research.

CHAPTER TWO

A Review of the Literature

2.1 Overview

The survey of the literature is in three parts. Part One begins by describing intuitions and (mis)conceptions in general, and the key literature in the area of probability. Researchers (e.g. Piaget, Fischbein etc.) in the fields of education and psychology have argued that students invoke both prior and intuitive information that facilitates, and sometimes hinders, their learning. Part Two focuses on the ideas of constructionism, in particular computer-based microworlds and situated abstraction. It describes what constructionism is and how it can be applied to mathematical learning; and it examines how situated abstractions can be made to connect concrete and abstract ideas, particular in the medium of computer-based learning environments. Finally, Part Three describes research on the learning of the concept of probability, and the didactical implications of this.

2.2 Part One: Intuitions and (Mis)Conceptions

2.2.1 What is intuition?

According to Fischbein (1975) intuition is

> 'the means by which intelligence secures for cognition an immediate control over action. An intuition is a stabilised program which is derived from experience, and which is effective because of its global, immediate, and flexible qualities. Deriving from action, it summarises, concentrates, and determines the anticipatory cognitive qualities of adaptive action in general, and of certain classes of adaptive actions, in particular' (p.20).

Fischbein's definition claims that intuitions develop as a kind of 'knowledge from experience', which are used to take control over actions. The work of Nunes, Schliemann and Carraher (1993) on 'street mathematics' reveals examples of mathematical intuitions developed by children. Following this definition, questions arise as to how intuitions that are based on experience can become 'stabilised programmes' when experiences change over time.

Bruner (1974) identifies two distinct approaches that operate in any field of intellectual endeavour. One is intuitive, the other analytic. In general, intuition is less rigorous with respect to proof, more oriented to the whole problem than to particular parts, less verbalized with respect to justification, and based upon a confidence to operate with sufficient data. Kant (1980) claims that intuition is simply the faculty through which objects are directly grasped in distinction to the faculty of understanding through which we achieve conceptual knowledge. In the area of mathematical thinking, Tall (1991) describes intuition as the product of the 'concept images' of the individual. The more educated the individual is in logical thinking, the more likely the individual's concept imagery will resonate with a logical response. The growth of thinking passes from initial intuitions based on pre-formal mathematics, to more refined formal intuitions as the learner's experience grows.

Fischbein (1987) points out that many other terms are used in reference to intuition. Sometimes people use the term 'insight' to indicate a sudden, global rearrangement of data in the mind, which provides a new view of a problem situation, or its solution. The terms revelation (especially in religious contexts), inspiration (in artistic matters), common sense, naive reasoning, empirical interpretation, natural thinking are sometimes also used to denote intuitive thinking. Fischbein states that intuitions serve several functions in the relation between actions and intellectual operations. They may be antecedent to the operations, or they may occur during the operations, facilitating their continuity and fluency. They may also occur after analytical operations, synthesising the results of analysis into a global view of unitary significance, and thus assisting in the transfer of decision to the level of action. For example, the individual, before being able to carry out any explicit computation of probabilities for a given situation, must adapt to an environment in which the accidental, the uncertain, and the possible are all part of ongoing existence.

Fischbein (1987) says that intuitions express a necessary mental capacity deeply rooted in our adaptive behaviour, as they reappear time and again in intellectual development. For example, over history, mathematicians have continued to discover that concepts that have previously been taken for granted as self-evident, have had to be questioned and sometimes abandoned. Fischbein suggests eight general characteristics of intuitive cognitions:

1. self evidence, as intuitive cognition is self-consistent, self-justifiable or self-explanatory,
2. intrinsic certainty: the fact that intuitive cognitions are accepted as certain,
3. perseverance: intuitions, once established, are very robust and sometimes the formal instruction has often very little impact on ones intuitive background,
4. coerciveness: intuitions impose themselves subjectively on the individual as unique representations or interpretations,
5. theoretical status: an intuition is a theory expressed in a particular representations using a model, but not a pure theory,
6. extrapolativeness: an intuition can be said to occur when an individual reaches a conclusion on the basis of less explicit information that is ordinarily required to reach that conclusion and always exceeds the data on hand,
7. globability: an intuition is a structured cognition which offers a unitary, global view of a certain situation,
8. implicitness: intuitive reactions are in fact the surface structure expression of tacit, subjacent processes and mechanisms.

Fischbein (1987) concludes that an intuition is the direct, cognitive prelude to action, mental or practical, which organises information in a behaviourally meaningful and intrinsically credible structure. Intuitions are self-evident notions that are robust, holistic, and conceptual, and to the individual, all of his or her intuitions are obviously correct. The robustness of an intuition becomes apparent by considering its applicability in many situations and the implausibility of alternatives. Furthermore, an intuition is not analytically separable into constituent parts but exists meaningfully as a whole. Building and applying an intuition requires conceptual thinking beyond immediate perceptual stimuli.

diSessa (1988) gives a definition of intuitiveness in relation to learning physics. He uses the term p-prims (phenomenological primitives) to describe the first abstractions from experience and he argues that intuitions consist of a number of fragments rather than one or even any small number of integrated structures one might call 'theories'. He states that:

> '... many of these fragments, which I call 'p-prims' (short for phenomenological primitives), can be understood as simple abstractions from

common experiences that are taken as relatively primitive in the sense that they generally need no explanation; they simply happen' (p.52).

It seems that diSessa has in mind a similar idea to Fischbein about the nature of intuition when he defines p-prims as 'knowledge that comes from experience'. Furthermore, diSessa describes them as a first meaning about things, consisting of simple abstractions, not well developed generalised pieces of knowledge that originate from specific experiences but can be used in similar situations. He emphasises p-prims as knowledge in pieces:

> 'People have perceptions about what happens, about what causes what, about what is important and what is not concerning knowledge, its development, and its deployment. In some cases these ideas also seem to be almost theoretical, but the same caveats are warranted here as with intuitive physics.' (p.67).

According to diSessa, *conceptual change* occurs in three ways, each involving a transformation of the 'causal net': a. by the addition of new p-prims, b. by the formation of new connections between p-prims and c. by changes to the priorities which fix how likely a p-prim is to be triggered by incoming data. 'Unstructured' p-prims live in isolation and cannot trigger further p-prims. They remain unstructured if incoming data is not inconsistent with those p-prims, or if there is no further incoming data. When p-prims and data are consistent with each other, p-prims become highly structured, so that a whole cluster is always triggered at the same time.

diSessa's idea about knowledge in pieces is connected to the theory of organisation of knowledge in 'schemata'. Piaget and Szeminska (1952) argue that infants construct intuitions when they abstract their behaviours into perceptual-motor schemas that allow them to recognise objects and events and to act appropriately. However, these authors do not explain how the process of creating early concepts occurs. It is here that cognitive psychology comes to fill the gap. Fischbein (1999a) proposes that intuitions change together with the entire adaptive system to which they belong. In the cognitive psychology literature (see for example Marshall, 1990), the adaptive system is described as a system of schemata.

Bruner (1966) discusses the idea of schemata. Highly specific action schemata that are irreversible guide sensori-motor intelligence: each action has its own plan, and the action goes off from beginning to end, and only in that order. Thus, human memory consists of networks of related pieces of information. Each network is a schema: a collection of well-connected facts, skills, strategies, and these schemata develop over long periods of time and by continual exposure to relevant contextual events. Fischbein (1999a) defines structural schemata as mental devices which make possible the assimilation and interpretation of information and adequate reactions to various stimuli. Structural schemata are characterized by their general relevance for adaptive behaviour. Fischbein and Schnarch (1996) state that:

> 'In each intuition considered, a general intellectual schema is embedded which influences the conclusion. The schema acts tacitly and this implies, in our opinion, that the schema becomes an integral part of the respective intuition. But, at request (when justification is required), the schema may be rendered explicit by the subject.' (p. 359).

The framework of schemata is proposed as fundamental to the organisation of the human mind and this it fundamentally shapes the mechanisms of learning. For example, Fischbein (1987) argues that there are sometimes contradictions between intuitive and scientifically acquired concepts, but the best procedure to make the student aware of the conflict is to help him/her develop control over his/her intuitions through conceptual schemas.

2.2.2 Intuitions in probability

For Fischbein (1999a), the renewed interest in intuition in the 20th century had two main sources: one was the continual endeavour of scientists to increase the degree of rigour, of conceptual purity, in their respective domains, and the other was their tendency to understand and explain the world as a whole by taking into account the genuine relativistic nature of physical laws.

In the area of probability, Kahneman and Tversky (1982) suggest that the term 'intuition' is used in three different senses. First, a judgement is called intuitive if it is reached by an informal and unstructured mode of reasoning, without the use of analytic methods or deliberate calculation. Second, a formal rule or a fact of nature is called intuitive if it is compatible with our model of world; thus, it is intuitively obvious that the probability of winning a lottery prize decreases with the number of tickets, but it is 'counter-intuitive' that there is a better than even chance that a sample of 23 people will include a pair of individuals with the same birthday. Third, a rule or procedure is said to be part of our repertoire of intuitions when we apply the rule or follow the procedure in our normal conduct. The rules of grammar, for example, are part of the intuitions of a native speaker.

Fischbein (1982) suggests that when subjects were required to predict the outcomes of a repetitive series of stochastic trials, they are able even from an early age, to tune the proportions of their predictions to the relative frequencies of the outcomes. It seems therefore that even very young children can have intuitions about relative frequencies. However, Fischbein (1982) explains:

> '...in order to create new correct probabilistic intuitions the learner must be actively involved in a process of performing change experiments, of guessing outcomes and evaluating chances, of confronting individual and mass results a priori calculated predictions, etc. New correct and powerful probabilistic intuitions cannot be produced by merely practising probabilistic formulae' (p.12).

Game playing is a very important activity for children's development of probabilistic ideas. Pratt (1998) points out that young children are very attracted to contexts where the laws of probability are central. Kafai, Franke, Ching and Shih (1998) have shown that game design provides motivation and engagement in ongoing reflection about the learning of mathematics. Most children love games based on dice or playing cards, where unpredictable events happen constantly. This gives us reason to believe that intuitions about stochastics do develop through experience. According to Fischbein (1975)

> 'probabilistic intuitions also involve images – images of dice, coins, boxes, and so on, but these images have a merely auxiliary function....The germ of intuitive reasoning about probability lies in natural 'experiments' with stochastic results, which involve predictions and random draws or other equivalent actions' (p. 16-17).

Fischbein argues that children, from pre-school up to the age of 7, can distinguish the 'random', in the sense of the unpredictable, from the deducible, however there are some general features of their intelligence at this age which distort their interpretations:

(a) Subjectivism: the child confuses the random with the arbitrary, interpreting the objectively random as the manifestation of the 'will' of the object concerned;

(b) Passive induction: the child judges new facts on the basis of the immediately preceding facts, and not on the basis of a deductive schema, a combinatory schema for example. This explains the inability of the pre-school child to correctly interpret random phenomena when the number of possible events is large;

(c) The belief that random events can be controlled by the person who triggers the events when, objectively, any such control is absent,

(d) The distinction between the random and the necessary is unstable in the absence of an operationally deductive system; inessential changes in the experimental conditions can easily influence the decisions of the child as to whether events are random or determined.

The probabilistic judgement of pre-school children is precarious. The interplay of influences at the cognitive level can easily mislead the child in making decisions, because of the absence of any relevant conceptual control. For example, Fischbein has studied the influence of perceptual configurations on the estimation of odds in pre-school children, demonstrating that children estimate odds and make predictions primarily on the basis of what they perceive.

Fischbein developed his theory out of Piaget and Inhelder's seminal research (1975). The latter's work contrasts with Fischbein's argument that probabilistic intuitions can become established given appropriate experiences – they claim that probabilistic thinking is a very late development in the child's evolution of knowledge. According to Piaget and Inhelder, the idea of chance is not acquired before the stage of concrete operations (before about the age of 7), because the understanding of chance presupposes an understanding of the irreversibility of a mixture of objects – for example the marbles used in one of the experiments of Piaget and Inhelder - and this requires the possession of a combinatory schema. In fact, the conceptual schema of chance can only exist as a function of the relevant operational resources. Fischbein (1975) argues contrary to Piaget and Inhelder, pointing out that primary intuition of chance is present in the everyday behaviour of the child, even before the age of 7. Chance is equivalent to unpredictability, and not necessarily to the smallness of odds; when the number of possibilities, and correspondingly the number of possible combinations, is small, the pre-school child reason correctly, and sometimes more correctly than an older child at the stage of formal operations.

Fischbein's (1975, 1987) theory of intuition and understanding classifies intuitions into primary and secondary types. Primary intuitions are formed before, and independently of, systematic instruction; they develop from normal everyday experience, which is of course subject to cultural variation. Secondary intuitions, by contrast, are formed after a systematic process of instruction; thus they are acquired, not through natural experience, but through some educational intervention. Secondary intuitions enable an individual to transcend primary cognitive acquisitions that are often inconsistent with the primary intuitions relating to the same concepts. They convey the products of social experience mostly in the form of scientific truths. The mathematician's intuitions, or those of the scientist, fall into this category. Fischbein (1987) argues that if our view of the close relationship between action and intuition is correct, it follows that secondary intuitions, even though attained through social experience, need to be reconstituted through individual experience, from which they are distilled.

What is not clear in Fischbein's account is how secondary intuitions are different from conceptions or whether they are the same thing. Is it right to call something an intuition which comes after a systematic process of instruction? And if not, why are secondary intuitions, still intuitions?

In the next sub-section I turn to the nature of concepts in order to understand better how secondary intuitions may relate to concepts.

2.2.3 (Mis)Conceptions

According to Smith, diSessa, and Rochelle (1993)

> 'Conceptions (or ideas) identify and relate factors that students use to explain intriguing or problematic phenomena. They also represent the knowledge, expressed in terms of solution strategies and their rationale, that constitutes the core solution to specific problems.'(p.119).

Having in mind this definition of conception, then a *misconception* is one that produces a systematic pattern of errors. Misconceptions are what people do wrong in opposition to logic; arise from students' prior learning, either in the classroom or from their interaction with the physical and social world. In domains like probability misconceptions continue to appear even after the correct approach has been taught. Tversky and Kahneman (1983) demonstrated this in the context of professional and common misconceptions about statistics and probability, such as the representativeness heuristic and the law of small numbers. According to Fischbein and Schnarch (1996), probabilistic misconceptions are rather divergent. Some probabilistic intuitions improve with age, but others become worse. This finding can be explained by the tacit influence of certain intellectual schemata on the structure of intuitions.

Tall and Vinner (1981) developed the idea of concept image and concept definition. They define concept image as that which describes the total cognitive structure that is associated with the concept, and concept definition as a form of words used to specify that concept. Vinner (1983) clarifies this idea, suggesting that for every concept there exists two different 'cells' in the cognitive structure. One cell is for the *definition of the concept* and the other is for the *concept image*. By concept definition, Vinner means a verbal definition that accurately explains the concept in a non-circular way. The concept image is a mental picture of a concept, and there are cases where the definition may be unclear, but the concept image is clear; for example, the definition for 'house' is less clear than its concept image. The word 'forest' can be defined as 'as many trees together' and we are able to visualise many trees together as a concept image. As Vinner suggests

> '...(1) in order to handle concepts one needs a concept image and not a concept definition, (2) concept definitions (where the concept was introduced by means of a definition) will remain inactive or even will be forgotten. In thinking, almost always the concept image will be evoked.'(p. 293).

One cell or even both of them might be void for a particular concept. There might be an interaction between the two cells although they can be formed independently.

Working from Vinner's description of concepts, it can be argued that secondary intuitions are more related with a concept image after instruction, than with the Fischbein's definition of intuition. Thus, it could be argued that Fischbein's secondary intuitions should be regarded as conceptions. In this study, I will use the term 'intuition' to denote a primary 'piece of knowledge' derived directly from experience, and I will relate secondary intuitions to Vinner's idea of concept image.

2.3 Part Two: Selected theoretical literature

This part is concerned with four theoretical issues, which form the theoretical framework of this study. These are: the theory of constructionism; the idea of situated abstraction as a mental construct which links concrete and abstract ideas in mathematics; computer-based microworlds in the field of mathematics education; and the role of representation and visualisation in learning mathematics.

2.3.1 Constructionism

Constructionism is a term first proposed by Papert (1990), seeking to combine the constructivist psychology of Piaget (e.g. 1952) and the ideas of progressive education exemplified by Dewey (1902). Constructionism emphasises that an effective way for the learner to construct knowledge in the head is to build something 'tangible', a meaningful product. According to Papert (1991)

> 'constructionism shares constructivism's connotation of learning as 'building knowledge structures' irrespective of the circumstances of the learning. It then adds the idea that this happens especially felicitously in a context where the learner is consciously engaged in constructing a public entity, whether it's a sand castle on the beach or a theory of the universe' (p.1).

The term 'constructionism' evokes and puts together two connotations: the psychological theory of constructivism and the notion of physical construction (e.g. a child's 'construction set' toy). Constructivism rejects the assumption that information can be simply passed on to a set of learners; it refers to a more complex procedure (see, e.g. Philips 1995). It recognises the learner as an active person, but it brings with it a tendency towards treating knowledge as entirely socio-political. Papert (1991) argues that, when Piaget described himself as a constructivist, he was referring to a view that knowledge structures are built by the subject rather than transmitted by a teacher. When he describes himself as a constructionist, Papert subscribes to this view, but he adds to it the idea that building knowledge structures happens especially well when the subject is engaged in building material structures in the world, as children do with construction sets. Children don't 'get ideas'; they make them by constructing something external and shareable, as Kafai and Resnick (1996) argue. The 'learning by doing' philosophy regards any transmission theory of education as fundamentally wrongheaded. Thus, children's learning can be promoted by

> '...using a cycle of internalisation of what is outside, then externalisation of what is inside and so on.' (Papert 1990, p. 3)

Over the last two decades, constructionism has gone beyond its original definition. Constructionist teachers and researchers have fleshed out this definition into a body of beliefs and practices, many of which are inspired by Papert's work, such as 'Mindstorms' (Papert, 1980). Whereas most learning theories describe knowledge acquisition in purely cognitive terms, constructionism sees an important role for affect. In constructionist learning, forming new relationships with knowledge is as important as forming new representations of knowledge. Constructionism also emphasises diversity: it recognises that learners can make connections with knowledge in many different ways, and they are more likely to become intellectually engaged when they are working on personally meaningful activities (Kafai and Resnick, 1996).

2.3.2 Situated abstraction: concreteness and abstraction

Papert (1996) argues that the metaphor of learning by constructing one's own knowledge has great rhetorical power against the metaphor of knowledge being transmitted though a pipeline from teacher to student. On a pragmatic level, constructionism suggests that we need to look for 'connections' as opportunities for learning, and on a theoretical level the metaphor leads to a range of interesting questions about the connectivity of knowledge. It even suggests that the deliberate part of leaning consists of making connections between mental entities that already exist. Also, thinking about the interconnectivity of knowledge can help to explain why some knowledge is so easily acquired without deliberate teaching.

Wilensky (1997) defines concretion as the building of personal connections, whereby learners express their own sense of experience across different mathematical domains. If we want to endow new ideas with suitable meaning, these should be connected with the learner's previous experience. Wilensky states that concreteness is not a property of an object, but rather a property of a person's relationship to an object - thus the formal often appears to be abstract just because we haven't yet constructed the connections that will concretise it.

The nature of abstraction, as a mental process and as an aspect of mathematical knowledge, has been discussed in mathematics education for many decades (see, for example, Noss and Hoyles 1996, for an overview of the issues). Looking for definitions of abstraction we find, for example, Gray and Tall's (2002), stating that an abstraction is a duality: it is both a process of 'drawing from' a situation and also the concept (the abstraction) produced by that process. Dreyfus (1991) argues that the process of abstraction is intimately linked to generalisation, the general nature of the results that can be obtained and passes also from the process of synthesizing. What we are used to think of abstraction is, I believe, best understood as what Sierpinska (2002) describes as a dual mental activity whereby some aspects of the object of thought are ignored while others are highlighted, or as Noss and

Hoyles (1996) describe, some conscious appreciation by learners of the generalised relationships implicit in their expressions.

This study adopts the notion of *situated abstraction* to describe a simultaneous notion of concretion and abstraction. Noss (2001) and Noss, Hoyles and Pozzi (2002) describe situated abstractions as a way of describing how a conceptualisation of mathematical knowledge can be simultaneously situated and abstract. The process cannot be separated from the product; there must be some 'webbing' (see Noss and Hoyles, 1996), connecting familiar knowledge to abstract ideas. Noss, Healy and Hoyles (1997) state that

> 'Our idea of *situated abstraction* is designed to underscore the idea that abstraction does not come ready-made, either a *priori* or *post hoc.* It is a process which develops in activity, which - like all activity - is situated'. (p.226)

They argue that abstracting within a domain of abstraction is situated in the sense that learners constructively generate mathematical ideas connected to the setting, articulated in terms of the objects and relationships within it - its tools, linguistic conventions and structures – generally speaking, the *medium of expression of the setting.*

Learners 'web' their own knowledge and understandings by actions in the learning environment (such as a microworld), articulating and 'messing with' fragments of that knowledge– this activity is abstracting within, not away from, the situation. Situated abstractions emerge during such activity as internal meanings, knowledge, concepts, that serve as relatively general devices for making sense of situations that arise within a setting. A situated abstraction is observable as a more or less tacitly articulated invariant relationship, framed within the situation itself.

Wilensky's (1997) idea of 'connected mathematics' suggests some didactical implications for situated abstractions. In connected mathematics, a concept cannot be intelligible if it has only one meaning - it is through connections that concepts gain meaning. Connected mathematics is a form of 'connected knowing'; a personal form of knowing that is intimate and contextualised as opposed to an alienated and disconnected formal knowing. Mathematical concepts derive their meaning and their power though their embeddedness in a personally and socially constructed web of connections to other ideas and experiences, both mathematical and non-mathematical. In a connected mathematics learning environment, the focus is on learner-owned investigative activities combined with reflection. Mathematical concepts are not simply taught as statements of formal definitions. Instead, they are multiplying represented in learning environments that support multiple styles and ways of knowing (an example of this multiple representation is given by Turkle and Papert, 1991). In general, connected mathematics calls for making many more connections between mathematics and the world at large, as well as between different mathematical domains, throughout the learning experience (Wilensky, 1997). Another expression of this kind of view of learning is Noss and Hoyles' (1996) idea of 'webbing':

> 'The idea of webbing is meant to convey the presence of a structure that learners can draw upon and reconstruct for support - in ways that they choose as appropriate for their struggle to construct meaning for some mathematics.' (p.108)

Webbing involves connecting together pieces the conceptual and physical worlds to produce understanding. The crucial idea is that individuals' sense of situation and the tools they have to hand provide support for making meaning, and also the means for reconstructing these pieces in new ways, or developing new pieces of knowledge. By webbing, concrete ideas become increasingly associated with abstract ideas. For an individual who has not had the opportunity, or does not yet possess the necessary internal meanings, a concept will be disconnected and unfamiliar. Thus, abstraction for Noss and Hoyles (1996) becomes a problem of how to add new friends and relations, not to ascend to unattainable heights.

Metaphorically, as Wilensky (1991) proposes, abstract objects are unreachable until concrete objects are used, which are reachable, 'graspable'. In these terms, concreteness is the property that measures the degree of relatedness to the object, how close we are to it, and the quality of our relationship with the object. The more the representation allows us to visualise an object, to pick out a particular scene or situation, the more concrete it is. Concreteness is associated with an instance of an object. It can be assumed that when we construct objects in the world, when we connect them with something already existing, we come into an engaged relationship with them and the 'abstract' knowledge needed for their construction - it is especially likely then that we will make this knowledge concrete.

Barnett and Noss (1997) have proposed that the process of making invisible mathematics visible might offer a more effective pedagogy for learning about probability. If children could gain an insight into making models of risk they would be able to connect their personal assessment of risk with statistical information that they are given. It can be said that the situated abstraction is the 'visible' side of mathematical understanding, before the idea gets its abstract view, and becomes generalised. Research has shown (for example Noss et al, 1997) that interaction with computer-based microworlds is important for developing situated abstractions in a constructionist environment. The following section introduces computer-based microworlds in detail.

2.3.3 Computer-based microworlds: situated stochastics

Balacheff and Kaput (1996) claim that a unique feature of effective computer-based material as compared to other types of learning materials is their intrinsically cognitive character. Similarly, diSessa (1995) suggests that by extending linear language into the multiply-connected, dynamic, richly textured graphical and interactive forms allowed by computers, we may fundamentally extend the material bases for thinking and learning, and with them the whole practice of education. diSessa (1986) argues that computers will not dominate our children's experience; they will play a part in it, or rather, many small parts. The trick, he says, is to turn abstractions into new experiences and not to turn experiences into abstractions with a computer. The following paragraphs will define what is a computer-based microworld, describe microworlds from Vygotskian perspective, elaborate microworlds for stochastics and illustrate microworlds and teaching.

2.3.3.1 Some definitions of microworlds

The key feature of a computer-based microworld is that it presents a formal, computable representation of mathematical objects and relationships. Balacheff and Kaput (1996) define a microworld as

> 'a set of primitive objects, elementary operations on these objects, and rules expressing the ways the operations can be performed and associated - which is the usual structure of a formal system in the mathematical sense and a domain of phenomenology that relates objects and actions on the underlying objects to the phenomena at the 'surface of the screen''. (p.471)

Similarly, Noss and Hoyles (1996) define a computer microworld as a flexible, interactive, expressive medium for working with mathematical objects and operations. Sutherland and Balacheff (1999) argue that computer-based microworlds provide access to formal mathematical knowledge through the nature of the 'intermediate' screen objects with which students interact in order to construct and manipulate new objects and relationships.

Moreover, mathematical microworlds allow the learner to explore simultaneously the structure of the accessible objects, their relations and the representations that makes them accessible. It can be said that the microworld 'evolves' as the learner's knowledge grows (Hoyles, 1993).

Papert (1980) offers the following description of the notion of a microworld, using the example of Turtle geometry and the 'Logo' programming language:

> ...the Turtle defines a self-contained world in which certain questions are relevant and others are not...this idea can be developed by constructing many such 'microworlds', each with its own set of assumptions and constrains. Children get to know what it is like to explore the properties of a chosen microworld undisturbed by extraneous questions. In doing so they learn to transfer habits of exploration from their personal lives to the formal domain of scientific theory construction' (p. 117)

Papert's early work has been very influential in the development of computer environments labelled 'microworlds'. Logo itself can be seen as a microworld for a particular kind of geometry. Learners interact with the microworld and build their own computer-based models. These models reflect learners' thinking about the mathematical objects and relationships as they work on particular activities.

The work done with Logo (described in, for example, Hoyles and Noss, 1992) provides an answer to Solomon's (1986) question for the future 'How do I learn to use computers today in a way that will not be obsolete in five years?' Solomon argues that children can use computers to gain concrete experience with dynamic processes acting separately or together in parallel. In her view the computer is an intellectual agent, operating in a culture and reflecting ideas of that culture. But what really inspires Solomon is the belief that naïve as well as sophisticated people should be given access to powerful ideas and that computers can offer easier access to such ideas as well as models for how to build with them. A computer-based microworld requires the learner to utilize formal structures in the service of more informal, intuitively based explorations and problem solving. Thus, microworlds are, according to Noss and Hoyles (1996), formal systems to which learners can relate to informally.

Edwards (1998) claims that in seeking a definition for 'microworlds', there are two possible approaches: the *structural* and the *functional*. The *structural* approach describes a microworld as: a. a set of computational objects, created to reflect the structure of mathematical entities within some sub domain of mathematics, b. a microworld links more than one representation of the underlying mathematical objects, c. often the objects and operations in a microworld can be combined to form more complex objects or operations, and d. a microworld includes a set of activities, which may be pre-programmed into the environment or instantiated in worksheets or verbal instructions in which the user is challenged to use the entities and operations to reach a goal. The *functional* view focuses on characteristics that emerge when a microworld is placed in front of a learner: the learner is expected to manipulate the objects and execute the operations instantiated in the microworld, with the purpose of inducing or discovering their properties, and construction and understanding of the system as a whole; to interpret feedback from these manipulations in order to self-correct or 'debug' his or her understanding of the domain; to use the objects and operation in the microworld to create new entities or to solve specific problems or challenges, or both.

2.3.3.2 Microworlds from a Vygotskian perspective

According to Lajoie, Jacobs and Lavigne (1995), computer-based learning environments support the 'learning by doing' philosophy; for example statistical concepts become less abstract if individuals interact directly with models rather than manipulate abstract symbols, which are detached from their referent. In a computational modelling approach to statistics, a modelling language and sets of associated tools are made available to learners, allowing the learner to pursue personally meaningful investigations. Arguing along similar lines, Harel and Papert (1990) describe 'instructional software' in which

> 'The communication between the software producers and their medium is dynamic. It requires constant goal-defining and redefining, planning and replanning, representing, building and rebuilding, blending, reorganizing, evaluating, modifying, and reflecting in similar senses.' (p. 46).

This view of designing software seems to be a realisation of Vygotsky's view of learning by an 'active child' in an 'active environment':

> 'An essential feature of learning is that it creates the zone of proximal development; that is learning awakens a variety of internal developmental processes that are able to operate only when the child is interacting with his environments and in cooperation with his peers. While these processes are

> intrernalized, they become part of the child independent developmental achievement.' (Vygotsky, 1978, p. 90)

According to Vygotsky (1981), all higher mental functioning that characterises human thought inherently involves mediation. A mediated mental function involves an indirect action on the world, which incorporates and transforms the natural, basic mental processes, extending their range and mode of functioning. The inclusion of the tool in the activity alters the course both of the activity and of all the mental processes that enter into the instrumental act. In this way, tools do not only facilitate mental processes, they transform, re-organise and shape them. Vygotsky (1978) argues that the child's system of activity is determined at each specific stage both by the child's degree of organic development and by his or her degree of mastery in the use of tools.

Noss and Hoyles (1996) add another 'brick' to Vygotsky's theory, connecting it with the use of computer-based microworlds. They argue that the computer environment acts as a window. That is, although one cannot observe a learner's thinking directly, the researcher can study and analyse the learner's actions as they 'come to the surface' whilst working with the situations that the computer environment provides.

diSessa, Hamer and Sherin (1991) adds another characteristic to Noss and Hoyles' (1996) openness of microworlds. They refer to the issue of ownership:

> 'Ownership of ideas and artefacts is a potential advantage to having students design representations. Did these children own and feel that they owned the ideas developed? At a finer scale, how was ownership shared in the group? Did individuals hang onto their own creations, adopt the group consensus, or adopt the ideas perceived to be best, independent of originator and independent of the feelings of the rest of the group?'(diSessa et al, 1991, p.124).

Children have in a computer-based environment the opportunity to express their own representations and also to re-construct and re-design the computer environment without losing its main kernel. This ownership helps to give meaning to the task and it enables children to feel that they own the ideas developed during their interaction with the task.

2.3.3.3 Microworlds for stochastics

Computational models can contribute to the learning of stochastics because, when learners build computational models of everyday phenomena, they can develop robust mental models of the underlying probability and statistics. The feedback provided by building and then testing the computational model supports the learner in debugging and successively refining their mental model (Wilensky, 1997). Learning about stochastics requires opportunities to inquire, investigate, analyse, and interpret rather than to compute and memorise (Papert 1980, 1996).

Research on constructionism and 'connected mathematics' has shown how technology empowers children in the use of stochastics (Wilensky, 1997; 1995; Pratt 1998). Wilensky's (1993) project on 'connected probability' is a detailed example of a computer-based microworld in the field of probability. The research aimed at a better understanding of the source of learners' difficulties in probability and statistics and the building of learning environments that would foster the development of intuitive conceptions of basic concepts, and positive attitudes towards the discipline. Wilensky's findings were that mathematical intuitions are not static, nor are some mathematical concepts inherently 'abstract' and thus not amenable to intuitive comprehension. Learners build and develop their mathematical intuitions over a lifetime and probabilistic ideas can become more concrete as learning progresses: the mathematical intuitions are constructed, not innately given. Both the lack of good learning environments for probability, and the cultural and epistemological confusion surrounding the subject are barriers to the construction of good probabilistic intuitions. Pratt's (1998) study showed that by using a constructionist computational system (Boxer), 9 – 11 year old children managed to make sense of local and global probabilistic meanings, interpreting local ones as those based on experiencing the outcome of individual events, and global ones as those that focus on an aggregated view of probability.

2.3.3.4 Microworlds and teaching

The above examples begin to suggest how the teaching process can be shaped by a computer-based constructionist microworld. Papert (1990) discusses 'constructionism vs. instructionsim' as follows

> ' This does not suggest that instruction is bad or useless. Instruction is not bad but overrated as the locus for significant change in education. *Better learning*

will not come from finding better ways for the teacher to instruct but from giving the learner better opportunities to construct'. (p.3).

As Papert (1991) argues, the classroom environment ought to change its focus from instruction to the idea of construction. Consistent with this view are two of the central tenets of constructionism: that people learn by doing, and by reflecting on what they do.

Learning by doing involves building up mental structures so that ideas may get linked into a mental network that will allow some ideas to assimilate readily while others will be transformed radically by the assimilating structure. If a concept is taught and that is well assimilated to teacher's internal structure but the structures of the learners are sufficiently different from the teacher's, then what is taught will be radically transformed (Wilensky, 1993).

Sutherland and Balacheff (1999) raise the issue of didactical complexity that can occur in computational environments for the learning of mathematics. They argue that computer-based microworlds can provide access to mathematical worlds but the very nature of mathematical knowing and knowledge means that for many pupils there will not be a seamless entry into the world of mathematics. This is where the teacher plays a crucial role, but the teacher needs to understand what has been passing between student, computer and the task. Moreover, Kapadia and Borovcnik (1991) claim that probability concepts and their meaning depend not only on the level of theory, but also on their representations. The tools used to represent knowledge or to deal with knowledge have a significant impact on an individual learner's formation of this knowledge. In particular, visualization and graphical methods facilitate representation of models at different levels of abstraction offering the possibility of interaction.

2.3.4 Representation and visualisation in mathematics

Constructionist learning environments encourage multiple learning styles and multiple *representations* of knowledge. In this section, some theories of representation and visualisation are described.

Bruner (1966) claims that " 'representation' must be inferred from the behaviour we can observe" (p. 7); to infer a person's representation of the world we must design tasks that permit us to infer how the learner does these things. Bruner describes three modes of representation: the enactive, the iconic, and the symbolic. In order to understand the nature of internal representation by action, image and symbol and the difficulties of inferring their presence-in-operation, we should first understand the *objective* of a representation — not simply the medium by which things are represented, but what they are represented for. Representation can be effected in the media of symbols, images, and actions and each form of representation can be specialized to aid symbolic manipulation, image, organization, or execution of motor acts. The three representational systems are parallel and each is unique, but all are capable of partial translation from one to another. Davis (1984) shares the idea of Bruner, adding that in order for any mathematical concept to be presented in the mind, it must be represented in some way, and the representations reveal the ideas one has of a specific concept.

Denis (1991) distinguishes two aspects of the representational process: *mental*, cognitive entities, and *external*, physical objects. Denis points out that a representation can refer to both a process and the outcome of this process, where the former is an activity generating objects or entities, and the latter refers to the entities themselves rather than the activity which produced them. The mental-external distinction of Denis is used in Dreyfus's (1993) analysis of representations, defining mental representations as those referring to a mathematical object or process (which might be different for different people), and external representations as referring to communication about mathematics, such as graphs. Booth and Thomas (2000) refer to cognitive integration, where mediation leads to the connection of imagery and related verbal data. They claim that to gain the benefits of cognitive integration it is necessary, for example, to have developed good spatial skills, in order to analyse a picture, abstract from it the essential elements of the problem, mentally convert these impressions of reality into theoretical objects, and at each stage relate these to the verbally stored conceptual ideas. Booth and Thomas's cognitive integration has similarities with Tall and Vinner's (1981) description of concept image and concept definition (see section 2.2.3). Dreyfus (1993) suggests that success in mathematics depends on rich mental representations, which involve many linked aspects of a given concept. He suggests that several mental representations of a concept may be presented simultaneously and be called up in different situation and also the learner can switch efficiently between them as required by the situation or problem with which he/she is faced.

Some representations are of visual form (e.g. a pie chart); others are purely symbolic or algebraic. The visual representation of a mathematical situation tends to provide a global view, while the symbolic representation tends to favour more local analysis (Larkin and Simon, 1987). The use of visual types of representation is called *visualisation* by Zimmermann and Cunningham (1991):

> ' We take the term visualization to describe the process of producing or using geometrical or graphical representations of mathematical concepts, principles or problems, whether hand drawn or computer generated' (p.1)

and they add

> ' In mathematics, visualization is not an end in itself but a means toward an end, which is understanding. Notice that, typically, one does not speak about visualizing a *diagram* but visualizing a *concept* or *problem*. To visualize a diagram means simply to form a mental image of the diagram, but to visualize a problem means to understand the problem in terms of a diagram or visual image. Mathematical visualization is the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding'. (p.3)

In fact, mathematicians' trust in visual reasoning has long been noted. Hadamard (1945), for example, illustrates that in problem solving mathematicians use visual reasoning, incorporating geometrical and other images as the basis for their intuitions, and only subsequently code them in symbolic terms. He recounts that when thinking mathematicians avoid not only the use of words, but also algebraic or other symbols and have a preference for vague images. Speaking of himself, he insists that words were totally absent from his mind when he really thought about mathematics, and that they remained absent until he came to the moment of communicating the results in a written or oral form.

Presmeg (1986) identifies five categories in visual imagery. *Concrete imagery*, a holistic image that has parts only to the extent that they are parts of an everyday object or picture; in Presmeg's study students tended to recognise the whole shape easily, but could not always recognise partially made shapes unless they are pictorial. *Pattern imagery* in which there is a conscious recognition of some of the properties of the concrete images and their relationships, often in the form of patterns, symbolic and numerical patterns. *Memory images of formulae,* where visualizers typically see a formula in their minds, written on a blackboard or in their notebooks. *Kinaesthetic imagery*, imagery involving muscular activity and *Dynamic imagery* in which shapes are changed into new related shapes.

Moreover, in Presmeg's study the students did not stay with only one of those types, but used different ones in different situations.

The range of visualizations generated by individuals is an important factor. Cunningham (1991) claims that adding visualization to mathematics education promotes intuition and understanding and allows a wider range of coverage of mathematical subjects. Students do not only learn mathematics but also learn new ways to think about and do their own mathematics. A particularly important way in which the mathematics education community has attempted to integrate visual reasoning is through the use of computer-based environments. Kaput, Noss and Hoyles (2002) describe how a computational environment can act as a 'representational window' and lead to the development of new notations for learnable mathematics. They claim that when one is learning or constructing something new, one needs to think explicitly about the representational system itself, the representational system is simultaneously transparent and opaque. This 'coordinated transparency' represents a synthesis of meaning and mechanism, a situation in which fluency with and within the medium can temporarily be replaced by a conscious awareness of its internal structures. Using 'ToonTalk' (a visual programming language for children), they illustrate how the evolution of representational structures and associated artefacts and technologies have gradually externalised aspects of knowledge and transformational skill.

The use of a computer-based microworld offers effective ways to represent stochastic ideas. Biehler (1991) summarises some main principles of using computers in the learning of probability: a. the computer should be used as a visualization tool; it adds new dimensions and the possibility to see pictures and dynamical representations, b. by the inclusion of multiple, linked representations; different students may find different representations persuasive and memorable, and all students will understand concepts more deeply if they comprehend the connections between representations, c. providing opportunities for interaction; students can modify data and graphs, set parameters for sampling experiments, and construct decision models. The immediate feedback of a visual representation provides students with the power to 'discover' statistical concepts. For example, Kaput (1995) creates environments that offer different representational 'windows' on the same general situation. The user can manipulate one of the particular representations simultaneously displayed and he argues that these experiences may help students link more familiar, concrete representations to more abstract ones of the same situation.

2.4 Part Three: Themes in research on the learning of probability

Probabilistic reasoning is receiving increased attention nowadays amongst mathematics educators and researchers. Psychologists have long been interested in how people reason about probabilities when they make decisions and they have amassed a considerable body of research in this area over the past few decades. This part is divided into two sections: the first illustrates the research in understanding probability; the second identifies some didactic implications arising from this research.

2.4.1 Research in understanding probability

The definitive texts on the development of probabilistic cognition were written by Piaget and Inhelder (1951, translated 1975) and Fischbein (1975). Unlike Piaget's preoccupation with 'a priori' probability, Fischbein's perspective allows an exploration of intuitive foundations and precursors to probabilistic knowledge. He is less interested than Piaget in the final schemata of formal probability that affect the outcome of instruction in the subject. There is a sense in which Fischbein is looking for the existence of partially formed probability concepts whereas Piaget is observing the lack of completed concepts.

Piaget and Inhelder (1975) devised a number of experiments, which involved 'chance' outcomes. They used these for 'probing' the conceptual development of children from pre-school ages to adolescence. The use of different experimental tasks with different age groups of course carries with it problems concerning the equivalence of the experimental situations, with consequent difficulties over establishing an unambiguous developmental picture. Piaget and Inhelder (1975) argued that for children before the age of 8 years old, there is a failure to understand the irreversibility of random mixing depends on the same reasons as the failure to understand the reversibility of operations. As they concluded for this age range:

> 'Perhaps it will be said that these last drawings show simply the lack of flexibility in operative representation and imagination at this age level, and do not exclude the understanding of physical mobility which the subject noted during the experiment itself. But if we compare these drawings with the totality of the reactions at this age, the over-all picture shows precisely that the lack of an internal mobility of thought goes along with failure to understand the mobility inherent in the physical process of random mixture itself'. (p. 12)

Piaget and Inhelder designed the 'marble tilt box' problem as a window on the concept of randomness. They assume that children who 'have the randomness concept' would interpret the situations as indeterminate and the transformations of the marble arrangements as random; and those who do not have the concept would anticipate conservation of the marble arrangement in its ordered state or, at least, a quick return to this special state. In the first stage of understanding, they claim that there is an absence of appreciation for the distribution of the whole and the subjects' reasoning is determined by two competing explanations: repetition or compensation. The subjects of this stage understand the mixture of the balls as a kind of regulated process of elements and because they lack precisely this understanding of the different combinations determining the individual paths of the balls during mixing, the subjects do not come to an understanding of any distribution of the whole based on the symmetry of combinations in play. Piaget and Inhelder described the second stage, until eleven years, as the stage of beginning the idea of combinatorics, where the intuition of chance appears along with the establishment of the first concrete operations ('interrelated' and 'reversible'). In the second stage there are beginnings of structuring a distribution and generalization from one experiment to the next. They claim that it is at the third stage, to twelve years, when formal thought first appears, that the process of random mixture is understood because, at that age, the observed facts are assimilated in an operative scheme based on the mechanics of permutations.

In general, Piaget and Inhelder inferred children's stages of development partly by observing their behavioural responses, which might, for example, be sketching a predicted outcome, and partly by questioning them on their predictions or preferences in sampling experiments. The connections between the outcome and the sample can be also characterised by 'causality'. As Piaget (1974) states:

> 'Since causality proceeds from a specific action to a generalization of relationships between the objects and, since the operations themselves are derived from actions and from their co ordinations, we can assume that the further back one goes, the more the actions of the subject are undifferentiated, therefore simultaneously preoperational and causal; as the operations progress, there will be at the same time differentiation and collaboration in a manner yet to be determined.' (p. 9)

According to Hawkins and Kapadia (1984) many experimental variations are possible using the framework of Piaget and Inhelder's research. In fact, there are so many variables, which may be altered either by accident or design, that apparently conflicting results occur in what essentially seem to be replication studies. Kuzmak and Gelman (1986) assess children's early competence in understanding probability and random phenomena, dealing with a random phenomenon as a physical one that is conventionally viewed as having a number of equally probable outcomes. They report that 4-year-olds differentially respond to the question of which colour is going to come out next (having as an option to answer 'yes' or 'no') when they are shown apparatuses with determined versus undetermined outcomes (a plastic tube containing a line of marbles, as opposed to a wire steel cage from which one marble will occasionally fall). They point out that children who fail to differentiate the phenomena do not tend to behave as if they view all phenomena as predictable, with a hidden and arbitrary order. They conclude that Piaget and Inhelder's characterization of an early stage of development was not confirmed and that the stage described by Fischbein, of having an intuition of uncertainty without a deep understanding of physical mechanism, may apply to the 4-year-olds who correctly differentiated the predictability of the two phenomena but could not give explanations showing an understanding of the random mechanism. However, they suggest that the 4-year-olds and perhaps even some 3-year-olds may have understood the nature of the mechanisms but may have lacked the linguistic ability to explain their thinking.

In fact, researchers (for example, Hoemann and Ross, 1971) have disagreed with Piaget's approaches, stating that his work is lacking in rigorous experimental controls, which enable unambiguous interpretations to be derived. This, and also disagreements over what kinds of probability concepts are being explored by Piaget's experiments, have engendered much debate and controversy. Green's (1983) investigations with large samples of children aged 7 to 16, using paper and pencil versions of Piaget's tasks, showed that ability to recognize randomness does not improve with age; the children were able to describe what was meant by equiprobable but they did not appear to understand the independence of the trials, and they tended to produce series in which runs of the same result were too short when compared to those that we would expect in a random process. Batanero and Serrano (1999) extended Green's research to 17 year-old students and complemented his results by an analysis of students' arguments to support randomness in bidimensional distributions. They conclude that students' arguments and responses indicate underlying conceptions that parallel some of the meanings attributed to randomness throughout history. These results reveal the complexity of the meaning of randomness and they argue that it may be preferable to consider the term randomness as a label with which we associate many concepts, such as experiment, event, sample space, or probability.

Fischbein's theory (1975) provides strong support for the importance of systematic education in the development of probabilistic intuitions. He provides an alternative to the developmental theory of Piaget and Inhelder and disagrees that the acquisition of probability concepts occurs in three narrowly defined stages, as Piaget and Inhelder describe. For Fischbein, the process of replacing a primary intuition by a secondary one, which occurs after instruction, is not a gradual process but takes place all at once. This is very much like the experience that Shaughnessy (1992) describes as the moment of discovery or insight in the problem-solving process. Fischbein also emphasizes the need in research on probabilistic cognition for more adequate analysis of the chance events under consideration and the task characteristics that may affect the participant's problem solving, as well as a more accurate identification of the cognitive processing that the participant actually uses.

In Fischbein's theory, intuition plays an essential part in the domain of probability, perhaps more conspicuously and strikingly than it does in other domains of mathematics. He argues that probability intuitions, for example probability matching whereby the relative frequencies of the person's predictions over a series of trials come to approximate the probabilities of the respective outcomes, have been observed in children as young as 3 – 4 years old and generally appears to be well established by the age of 6. Fischbein (1999b) emphasizes the role of intuition in the origins and development of probabilistic thinking:

> 'If one investigates the student's difficulties and misconceptions, one does not identify only logical deficiencies. One identifies, very often, intuitive tendencies, intuitive interpretations, and models tacit or conscious - that contradict the formal knowledge with which school tries to endow the student.'
> (p. 49)

Research carried out by Kahneman and Tversky (1982) into the persistent errors that people make when making judgements under uncertainty, shows a difference between looking at intuitions and looking at misconceptions. It also shows that because of the contradictions between intuitions and the formal knowledge that school tries to teach, this can cause students to develop many different misconceptions. Kahneman and Tversky's original thesis was that people who are statistically naïve make estimates for the likelihood of events by using certain judgmental heuristics, such as 'representativeness' and 'availability'. According to the representativeness heuristic, people estimate the likelihood of events based on how well an outcome represents some aspect of its parent population; people believe that a sample should either reflect the distribution of the parent population, or that a sample should mirror the process by which random events is generated. For example, Tversky and Kahneman (1983) analysed reactions to the following statement:

> 'Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.'(p. 297)

People had to rank by likelihood the statements 'Linda is a bank teller', 'Linda is active in the feminist movement', and 'Linda is a bank teller and is active in the feminist movement'. A misconception that respondents exhibited was to rank Linda's being a bank teller and a feminist as more likely than just being a banker. This ranking violates the rules of logic, which require that any single statement is more likely than its conjunction with another statement.

Kahneman and Tversky (1982) concluded that people have many misconceptions of probability events. Naïve reasoners are posited to have what might be called a 'makes sense' epistemology, they act as though the test of truth is that a proposition makes intuitive sense - it 'sounds right', 'rings true'. They see no need to criticise or revise accounts that do make sense because the intuitive feel of fit suffices. Stochastic misconceptions have been seen when:

a. people inappropriately believe there is no variability in the 'real world',

b. people have unwarranted confidence in small samples,

c. they are unaware of regression to the mean in their everyday lives,

d. they mistakenly believe that an appropriate size for a random sample is independent of the overall population size,

e. people believe that any difference in the means between two groups is significant,

f. people have insufficient respect for small differences in large samples

(Tversky and Kahneman 1983; Schoenfeld, 1985; Shaughnessy, 1985).

Tversky and Kahnheman (1983) claim that the law of small numbers reflects a failure to appreciate the chance and variability in small sets drawn from a population. In research on understanding and attribution of randomness of primary grade children, kindergartners and 3rd graders, and undergraduates, Metz (1998) suggests that the law of small numbers was the only interpretation to explain the subjects' acknowledgment of determinism. That is, the belief that the contents of an unknown sample space can be directly assessed through a small number of observations of the constituent elements. She claims that the challenge of assessing the boundaries of the agent's control enters into the instantiation of the strategy, in which the participant believes he or she should somehow be able to implement the drawing process such that the order of marbles drawn accurately reflects the proportion in colours in the unknown sample space.

The misconceptions research, such as that of Kahneman and Tversky (1973), argues that in making predictions and judgements under uncertainty, people do not appear to follow the calculus of chance or statistical theory of prediction. Instead, they rely on a limited number of heuristics, which sometimes yield reasonable judgements and sometimes lead to systematic errors. On the other hand, Smith, diSessa and Rochelle (1993) refute the misconceptions approach as lacking explanatory power: this approach only considers a narrow range of contexts in which the misconceptions occur. Heuristics research is full of questions directed to people for whom those questions are clearly not close to their areas of competence or in situations where appropriate tools to explore the questions are unavailable. Smith, diSessa and Rochelle (1993) state that it should be possible to create new learning environments, which are closer to pre-existing areas of competence of young children, in which it may be possible to observe their search for underlying principles. To understand children's physical conceptions they claim that

> 'It seems more productive to study the roles that naïve physical conceptions continue to play in expert reasoning than to suggest that the main issue in acquiring expertise is to remove and replace them.' (p.145).

Similarly, such an approach may provide a richer evidential base for understanding probabilities than an emphasis on how people fail when operating outside of their area of competence.

Hawkins and Hawkins (1997) conducted research into lawyers' misconceptions in the area of statistics and probability. Lawyers in the United Kingdom receive no training in these areas apart from their compulsory school mathematical education. The researchers concluded that it is not surprising that lawyers have considerable difficulty with most of their probabilistic questions, given the inadequacy of their preparation for the increasingly quantitative decision-making facing them in their work.

The literature that describes misconceptions in statistics and probability is much more extensive than the literature on what can practically be done to ameliorate them. Wilensky (1995, 1997) is critical of the influence of Kahneman and Tversky and the consequent belief that humans are incapable of thinking intuitively about probability. He argues that a common conclusion drawn by educators and researchers from this research is that people just aren't built for doing probability, or that intuitions are faulty and not to be trusted. He claims that a better practice for educators wishing to educate students in probability is to instil an attitude of mistrust for intuitive responses and a healthy respect for formulae. Wilensky's research showed that when people have difficulties, or 'misconceptions' according to Kahneman and Tversky, these stem from fundamental confusion about notions such as randomness, distribution, and expectation, and these can be avoided by using an appropriate learning environment (in Wilensky's case the 'connected probability' project.

Jacobs and Potenza (1991) found that the use of the representativeness heuristic is specific to social judgements and is based on the development of social schemata that can be used to make judgements in social situations. Nisbett, Krantz, Jepson and Kunda (1983) suggest that people are disinclined to reason statistically about certain kinds of events that they recognise to be highly variable and uncertain, notably social events, because the sample spaces for the events and the chance factors influencing the events are opaque. They add that training in statistics should promote statistical reasoning even about mundane events of everyday life, because such training can help people to construct distributional models for events and help them to recognise 'error', or the chance factors that influence events.

Konold (1989) suggests that in addition to normative, formal reasoning and to reasoning with heuristics, people also reason in some situations according to an 'outcome approach'. They are inclined to view probability inappropriately as 'operative', i.e. as attempting to predict the outcome of an event. Given an uncertain situation, people using the outcome approach do not see their goal as specifying probabilities that reflect the distribution of occurrences in a sample but as predicting the results of a single trial. Konold found that subjects' responses were not consistent across problems: subjects who appeared to reason according to the outcome approach on one problem seemed to reason correctly on another and showed evidence of using a heuristic approach on yet a third. He concludes that there is a tendency for people to interpret the use of 'probabilities' in certain situations as measures of causal dependence, rather than as a measure of chance. Li and Pereira-Mendoza (2002) conclude that the outcome approach is one of the main 'misconceptions' of probability in Chinese students, independent of school streams or background in probability, and also that students' understanding of probability does not improve naturally with age, although teaching does play an important role.

Konold, Pollatsek, Well, Lohmeier and Lipson (1993) concluded that most students have a well-developed concept of independence prior to any formal instruction. The sizeable percentage of correct responses are spurious and reflect an outcome approach to uncertainty that is perhaps more pernicious than misapplication of the representativeness heuristic. They state that the picture that is emerging from research on student conceptions of probability is that there is no simple story about how students reason about chance and it is important for teachers of probability to become familiar with the variety of alternative conceptions.

Fischbein, Nello and Marino (1991) tried to obtain a better understanding of the origins and nature of some probabilistic intuitive obstacles of elementary and junior-high-school. They found that subjects did not have in mind a clear definition of the terms 'possible', 'impossible' and 'certain'. The term 'certain', especially, entails a difficulty when it is related to a compound event, for instance the probability of obtaining a number smaller than 7 when rolling a die. Some subjects confused 'rare' with 'impossible'. They also conclude that there is no natural understanding of the fact that, in a sample space, possible outcomes should be distinguished and counted separately if the order of their elementary components is different. In questions where the subjects were asked to compare the probabilities of getting certain numbers obtained by addition when rolling two dice, they found that many subjects seem to be able to relate spontaneously the estimations of probabilities to the magnitude of sample spaces. Singer and Resnick (1992) suggest that proportions expressed as relationships have three basic quantities associated with them, a whole and two parts, and that they can be represented by two different schemas: a part-whole schema and a part-part schema. For example, in a collection of ten marbles, six of which are red and four of which are black, one can represent the relationship of the parts to the whole (4/10 black to whole) or the relationship of the two parts to each other (6/4 red to black).

Fischbein et al (1991) claim that

> 'The general idea is then that, the outcomes *can be controlled* by the individual. The mathematical, probabilistic structure has not yet been detached

> from the concrete circumstances and considered in its abstract generality' (p. 530).

Truran and Truran (1999) studied the understandings of the concept of independence of dice throws by children and adults, showing that while some probabilistic intuitions exist at a very early age, they do not transfer well into even moderately complicated situations. The suggestion from Konold (1989) that it is a lack of understanding of independence, which is the principal problem, becomes more precisely that it is a failure to know how to identify random generators. For Truran and Truran (1999), the principal problem seems to be a belief that the initial tossing of the die, the moment that the action of tossing takes place, is more powerful than the subsequent large number of forces that impinge on the random generator. They conclude that since deciding whether two random generators are independent is essentially a subjective process and since we can see that many children and adults do not always intuitively acquire this skill, it is something that needs more explicit attention in schools.

Jones et al (1997; 1999) try to give a framework for assessing probabilistic thinking. They conclude that there are four levels for each of the following probabilistic concepts: sample space, probability of an event, probability comparisons and conditional probability. They describe the four levels as follows:

Level 1- 'subjective', children list an incomplete set of outcomes for a one-stage experiment; predict the most/least likely event based on subjective judgements, recognise certain and impossible events; compare the probability of an event in two different sample spaces, usually based on various subjective or numeric judgements, cannot distinguish 'fair' probability situations from 'unfair' ones, following a particular outcome will predict consistently that it will occur next time, or alternatively that it will not occur again (over generalisation).

Level 2- 'transitional', children list a complete set of outcomes for one-stage experiments and sometimes list a complete set of outcomes for a two-stage experiment using limited and unsystematic strategies, predict most/least likely event based on quantitative judgements but may revert to subjective judgements, make probability comparisons based on quantitative judgements-may not quantify correctly and may have limitations where non-contiguous events are involved-, begin to distinguish 'fair' probability situations from 'unfair' ones, begin to recognise that the probability of an event changes in a non-replacement situation and can recognize when certain and impossible events will arise in non-replacement situations.

Level 3- 'informal quantitative', children adopt and partially apply a generative strategy to make a complete listing of outcomes for a two-stage case, predict most/least likely events based on quantitative judgements including situations involving non-contiguous outcomes, use numbers informally to compare probabilities, distinguish 'certain', impossible', and 'possible' events, and justify choice quantitatively, make probability comparisons based on consistent quantitative judgements, justify with valid quantitative reasoning, but may have limitations where non-contiguous events are involved, distinguish 'fair' and 'unfair' probability generators based on valid numerical reasoning, can determine changing probability measurers in anon-replacement situation and recognise that the probability of all events change in a non-replacement situation.

Level 4- 'numerical', children adopt and apply a generative strategy which enables a complete listing of the outcomes for two-and three- stage cases, predict most/least likely events for single stage experiments; in this level children also assign a numerical probability to an event (it may be real probability or a form of odds), assign a numerical probability measure and compare, incorporate non-contiguous and contiguous outcomes in determining probabilities, assign equal numerical probabilities to equally likely events, assign equal numerical probabilities in replacements and non-replacement situations and distinguish dependent and independent events.

Jones et al's framework suggests that the development of young's children's thinking in probability will be linear - in order for children to reach a 'level 3' understanding they must pass first from level 1 to level 2 etc. The subjects of Jones et al's research were eight randomly selected children; four from each of two third-grade classes, and none of the children had been exposed to probability instruction. This framework seems to be the only recent one, since Piaget and Inhelder (1975), which tries to define stages for children's development in the concepts of probability.

Concerning understanding of sample space, Ayres and Way (1999; 2000) worked with students who observed a video recording of coloured balls being drawn from a box with replacement, sample space unknown. They conclude that students were significantly influenced in their probability judgements by confirmation or refutation of their own 'predictions', and that many students inappropriately tried to utilise colour patterns as a strategy.

Falk et al (1980) suggest that one method to help develop young children's potential for understanding of probability is to let them practice playing probability games, which give children experience with the operation of the laws of probability. Paparistodemou and Philippou (2002) describe how young children start to make probabilistic decisions and think about chance and risk from an early age, depending on *how* they have embarked on probabilistic games. Amir and Williams (1994) argue that heuristic intuitions seem to gain for relevant experiences and practices, which are culturally determined, like gambling or board games. Playing games, using concrete materials and 'real life' tools is a kind of concrete, simple representation that can be a force for learning probability (Chiu, 1996; Acredolo et al, 1989; Szendrei, 1996). Watson and Moritz (2000) argue, from research on statistical literacy, that there is a need to make more explicit the transition from out-of-school connotations of sampling, such as in supermarkets or medical contexts, in which variation is not usually an issue, to the representative sampling required for statistical inference.

In research on fairness, Pratt and Noss (1998) examined how their subjects, 9 – 11 year old children, made sense of dice situations, and they showed how existing intuitions about fairness, often based on actual outcomes, are co-ordinated with new meanings and derive from interacting with a computer-based microworld. They permit a data-oriented view of the world, where Kahneman and Tversky's heuristics and Konold's outcome approach can flourish. These ways are abstracted directly from experience with dice and other kinds of random generators, like cards, during informal game playing. This is also what Papert (1996) argues: learning probability by throwing dice and calculating fractions will reinforce behaviour based on misconceptions such as Kahneman and Tversky's identify. Borovcnik and Peard (1996) indicate the gap between actions (or operations) and reflections. For them, an operation means for example 'predicting the outcome of the next toss', whereas reflection means 'evaluating the weight of heads'. Thus, the individual experiences a conflict right from the beginning. This conflict is governed by, on the one hand, being unable to predict the next outcome exactly and with absolute certainty, and on the other hand the need to master the 'chaos' in the environment.

This is where the computer can play a role, as Biehler (1991) points out, for example in dealing with the law of large numbers and the frequentist interpretation. Biehler argues that students often understand the laws of large numbers only superficially. Without computer support, it is in fact difficult to work with large numbers; long run frequencies remain mysterious and students are led to act as if a 'law of small numbers' holds - they develop the 'misconception' that there is a stabilization of absolute frequencies. Aspinwall and Tarr (2001) report that, while typical middle school students are seemingly unaware of the relationship between experimental probability and sample size, appropriate cognitive activity focused on results of simulations of random phenomena can foster conceptual development. In general, the relationship between experimental probability of an event and the theoretical probability of an event results from the fact that, for a given event, experimental probability will more closely approximate theoretical probability as the number of trial increases. Stohl and Tarr's (2002) research has shown how a variety of software tools can enable students to understand the interplay between empirical and theoretical probability. Their study was based on 6th grade students who worked with open-ended software stimulation tools, using a program, Probability Explorer, as the primary investigation tool. The students began to recognize the importance of using larger samples to make inferences, and to justify their claims with data based evidence. These authors claim that one of the most important aspects of formulating and evaluating inferences is understanding the unpredictability of random phenomenon in the short-run and predictability in the long-run trends in data (i.e., the law of large numbers). Thus, the sample size in a simulation is a crucial factor for students to consider when making inferences from a sample distribution to the population, and making connections between empirical and theoretical probability. The terms short-run and long-run trends here are used in a similar sense to Pratt's (1998) local and global meaning. Stohl and Tarr (2002) suggest that inference is an appropriate topic for the middle school mathematics curriculum and it does not need to be postponed until students have first developed robust proportional reasoning. This brings us to the next section of this chapter, which describes the concept of probability in curriculum and in teaching procedures.

2.4.2 Didactic implications concerning the concept of probability

According to Borovcnik and Peard (1996), there is no doubt that the topic of probability is an important one in the mathematics curriculum even though the inclusion of probability is a relatively recent development. A necessary area of investigation is the role of the teacher: if we consider the necessity of educating students who are used to think stochastically, it is needed to re-think the role of the teacher in the teaching/learning process (Lopes and de Moura, 2002). Konold (1991) provides some salient reflections on the role of instruction in a learning situation fraught with students' prior stochastic' misconceptions':

> 'Long before their formal introduction to probability, students have dealt with countless situations involving uncertainty and have learned to use words such as probable, random, independent, lucky, chance, fair, unlikely. They have a coherent understanding that permits them to utter sentences using these words that are comprehensible to others in everyday situations. It is into this web of meanings that students attempt to integrate and thus make sense of their classroom experience… My assumption is that students have intuitions about probability, and that they can't check these in at the classroom door. The success of the teacher depends on how these notions are treated in relation to those the teacher would like the student to acquire... How students think about probability before and during instruction can facilitate communication between the student and the teacher' (p.144).

Borovcnik and Bentz (1991) suggest that conventional teaching establishes too few links between primary intuitions and the mathematical model. They claim that this is critical for probability as there are no direct experiences, which will help learners to establish these links on their own. If learners are to understand probability and apply it to other situations, teaching has to start from the learner's intuitions and develop them gradually. That is, the learner needs to see how mathematics can helpfully reconstruct his/her intuitions. Konold (1989) suggests that intuitions and misconceptions need to be considered in the design both of probability curricula and of instruments meant to assess conceptual understanding

Jones, Langrall, Thornton and Nisbet (2002) indicate that a considerable amount of research has been concerned with young children's probabilistic thinking, yet probability is an under-represented mathematical domain in elementary school curricula. The research of Kahneman and Tversky (1982) did inspire curricula to alert students to the use of heuristics and how these heuristics can lead them astray in judgements of uncertainty.

One of the issues for curriculum design is the continuing influence of Piagetian models for the development of intelligence. For example, Ojeda (1999) claims that probability is not considered in Mexico for children of 5 to 8 years of age, because, according to Piaget, children are not capable to understand probability at stages previous to the stage of concrete operations. However, Ojeda's research suggests the need to introducing didactical activities for teaching probability at elementary school level, by giving priority to a natural approach that recognises children's intuitive understandings of chance events.

Ahlgren and Garfield (1991) suggest what a probabilistic curriculum should involve. One should seek to ensure that students will be able apply probabilistic understandings outside of the school setting, but an impediment to meaningful curriculum is the tendency by students to compartmentalise what they learn in school. Without practice in retrieving ideas or employing skills outside of the classroom context, students are unlikely to make use of what they learn in school. This tendency is even stronger for ideas that are inconsistent with their intuitive ways of viewing the world. Borovcnick and Bentz (1991) also suggest that from the didactic perspective it may be advantageous to develop concepts, which allow for a more direct development of stable secondary intuitions than for probability concepts.

Steinbring (1991) argues that there are two main methods of teaching probability, based on intuitiveness and consistency. The intuitive approach focuses on experiments, ideal games of chance and real situations. After this intuitive and experimental phase of teaching there is a progression towards statistical methods and concepts. The teaching of probability and statistics generally relies heavily on the concepts of chance and randomness, therefore statistical educators should be aware of the 'theories' and preconceptions concerning these concepts that students possess before receiving any instruction, since while students are learning something new, they will construct their own meanings by connecting the new information to what they already believe to be true (Falk, 1992; Falk and Konold, 1994; Borovcnick and Peard, 1996; Batanero, Serrano and Garfield, 1996).

In research on the evolution with age of probabilistic intuitively based misconceptions, Fischbein and Schnarch (1997) conclude that some misconceptions diminish with age, some remain stable and some gain greater influence. Their interpretation of this is that probability does not consist of mere technical knowledge and procedures leading to solutions. Rather, it requires a way of thinking that is genuinely different from that required by most mathematics learnt in school. In learning probability, students need to develop new intuitions, and instruction should lead students to actively experience the conflicts between their primary intuitive schemata and the particular types of reasoning specific to stochastic situations.

Wilensky's (1995) research aimed to build from the conjecture that both the learner's own sense-making and the cognitive researchers' investigations of this sense-making are best advanced by having the learner build computational models of probabilistic phenomena, based on prior intuitive understandings. He shows that through such building, learners can come to make sense of core concepts in probability, like normal distribution. Similarly, Pratt and Noss (1996) describe the development of a computer-based domain within which children (aged 9 - 11 years) manipulate 'stochastic gadgets', representing everyday objects such as a die, a coin, a lottery and a set of playing cards. They claim that this gives researchers a 'window' onto the processes by which the domain shapes the children's thinking about stochastic events such as fairness, randomness and chance. Pratt and Noss put individual learners in situations where they could express their beliefs in symbolic ('programming') form, where they could articulate the beliefs that they hold, and reconstruct them in the light of their experiences. Pratt's (1998; 2000) research on computational environments examines two notions: a. that probability is 'simply hard' and b. that our knowledge of how to build effective learning environments is too limited. As he states, the evidence from his study is that pedagogic methods need to be found by which recently acquired global meanings can be recognised to have greater explanatory power than competing long-established local meanings. According to Pratt (1998), local meanings in probability have characteristics such as: the next outcome is unpredictable, there is irregularity, there are no patterns and fairness appears as symmetry in appearance. Global meanings have characteristics such as: the proportion of outcomes for each possibility is predictable (probability), the proportion will stabilise as an increasing number of results (large numbers) and there is control through the manipulation of sample space.

Greer (2001) argues that particular attention needs to be given to the relationship between probability and statistics and how this relationship should be handled instructionally. The recording, graphical representation and interpretation of results from experiments with stochastic phenomena, in combination with comparisons of graphical data arising from probabilistic and deterministic processes (and the fitting of algebraic functions), offer opportunities for making such links. Petocz and Reid (2002) indicate the importance for development of learning environments that can engage students' interest, broaden their understanding of statistics and enrich their own lives. They suggest that the development of learning environments must be 'total' and that the learning of stochastics should be less focused on the curriculum itself, and certainly less focused on the traditional concern of material to be 'covered' or 'examined'. Rather, the focus should move towards supporting students to develop 'holistically'.

2.5 Summary of Chapter Two and the emerging focus of this study

The review of the literature reveals that there is a gap between intuitions and construction of the mathematical concepts of probability. Some researchers have suggested that this gap can be explained in terms of 'misconceptions', or 'wrong connections' between pieces of probabilistic knowledge. However, despite the important contributions that Piaget and Inhelder, Tversky and Kahneman and other similar works have made to the research of randomness, the research has paid rather scant attention to the tools that people have available for expressing ideas about randomness, fairness and more generally, probability. Tools that represent knowledge or allow learners to 'manipulate' knowledge have a sizeable impact on learners' formation of this knowledge. In contrast to the Piagetian research, constructionism has emphasised that an effective way for the learner to construct knowledge in the head is to build something 'tangible', a meaningful product. Constructionist learning environments encourage multiple learning styles and multiple representations of knowledge. In the constructionist paradigm, using tools that are specially designed for expressing randomness and chance, learners can express ideas that cannot be predicted simply by misconceptions or by stages of thinking.

This study takes the framework of constructionism as a working hypothesis, that when learners construct for themselves they will express ideas in different ways. Thus, this study tries to develop a tool-based game with which children can express their probabilistic intuitions. In terms of analysing children's work in a constructionist environment, the notion of situated abstraction will be the principle to describe the movement from 'concrete thinking' to abstraction. After the design and evaluation of the tool-based game, the study will focus on analysing children's expressions of chance events. The next chapter will provide an overview of the aims and the elements of the game.

CHAPTER THREE

Aims of the Study and Overview of the Game

3.1 Overview

This chapter provides an overview of the framework of the study. I begin this chapter by stating the aims of the study and then I discuss the framework within which the study took place. Specifically, the structure and the design principles of the tool-based game will be discussed and based on that, the aims will be restated. The idea here is for the reader to gain an overview of the pieces that comprise this research.

3.2 Aims

The general theme that ran throughout this study was to explore the ways in which a specially designed computer game afforded children the opportunity to develop and express probabilistic ideas. Specifically, the aims of the study were:

Aim 1: iteratively to design a tool-based game to afford young children (age 5½ -8) opportunities and novel ways to express and develop probabilistic ideas; and

Aim 2: to describe and analyse how the tool-based game mediated the children's expression of chance events.

By the idea of "game" (the relationship between games and intuitions are explained in section 2.2.2) I intend something special, which is not necessarily part of what is normally meant by the use of the word 'game'. In normal speech a game is something that is merely played for enjoyment. But in this study, the game was devised not only for playing, but also for understanding. The key was to render *visible* its structures and the mathematical ideas that underpinned it. In this respect, the game was "open" whose rules were visible. Henceforth, I will drop the adjective tool-based in front of the word 'game', although it is essential that the reader insert it for his/her self.

The game format facilitated the detailed tracing of children's initial knowledge and intuitions, because the activity within the computational medium required the articulation of these pieces of knowledge by the 'children players'. The previous chapter illustrates evidence that using a computer game to express and explore probabilistic ideas is feasible in contexts that are based on visual manipulations and which offer access to formal ideas in a concrete way, by representing abstract mathematical ideas in iconic form on the screen.

The age of children who participated on the study was between 5½ to 8 years old and they were generally characterised by their teachers as normal mathematical ability children of both sexes. The choice of this age range was based on three practical and theoretical considerations:

1. The literature review showed that little research on probability has been conducted with children at pre-primary and at the early-age of primary school. Most research in the field of probability has been undertaken with secondary school pupils (or with late elementary school ages).
2. A second reason for choosing this age range was curriculum-based. The National Curriculum for England and Wales, and also that of Cyprus, contains almost no work on chance and probability at this level. This lack benefited the research in making it assumable that at this age formal probabilistic knowledge is still undeveloped and intuitive knowledge plays the major role.
3. What little research has been undertaken concerning probability at the early-age of primary school has tended to focus on the existence of probabilistic stages/levels. This research has sought to examine how far probabilistic thinking follows linear stages at this age and to pay attention to the tools that children have available for expressing probabilistic ideas. It adopted the constructionist paradigm approach that when children have available the tools to express themselves they do extraordinary things that cannot be predicted by stages of thinking. This research seeks into describe and analyse how the game mediated children's expressions of chance events and to compare these mediations with probabilistic thinking frameworks (for example Piaget and Inhelder, 1975; Jones et al, 1997).

The study was divided into two phases: each corresponding to one of the two main aims of the study. Phase 1 deals with the iterative design and evaluation of the game. The methodology of this work is reported in Chapter 4 and the evolution of the game is reported in Chapter 5. Phase 2 investigates the way in which children learned and describes in detail the expressed ideas of children by using the game. The methodology of this work is also reported in Chapter 4 and the analysis of the children's expressions of chance events by using the medium is described in Chapter 6,7,8.

I begin by outlining the general structure of the game.

3.3 The structure of the game

In Phase 1 of the study, the aim was to design a game that afforded young children an opportunity to express probabilistic ideas. The game was based on icon based programming environments. Each of these programming environments was built in a new version of Logo, called "Imagine", that was developed concurrently with the study. Imagine and its programming environments were developed through a European Union Financed Project, Playground Project[1]. The aim of the Playground Project was to design and build iconic rule-based programming system. The project team was building computer environments for 4-8 year-olds to play, design and create games. Playground Project's rationale was for children to build and modify games by constructing and expressing their ideas with rules. The project aimed to harness children's playfulness, allowing them to enter into abstract and formal ways of thinking. The relation between Imagine software, its programming environments (platforms) and the games used in each Phase is illustrated in Diagram 3.1.

[1] http://www.ioe.ac.uk/playground/frame_f.htm

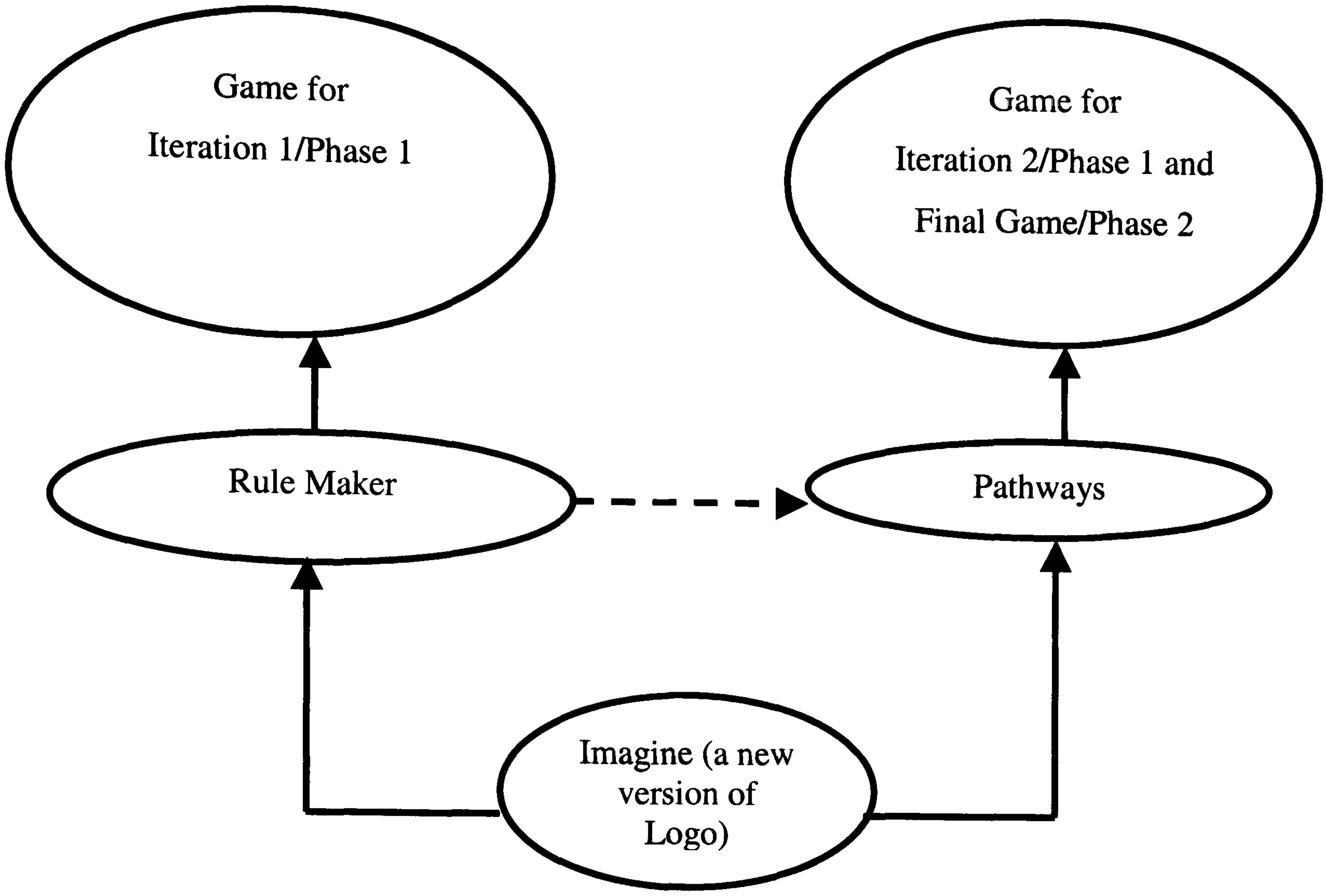

Diagram 3.1: The relation between the Imagine software and its two platforms, the relation between the two platforms and the relation between the platforms and the games used for each Phase

Diagram 3.1 describes the structure of the game. The programming 'base' of designing the games was 'Imagine' in which the platforms Rule-Maker and Pathways were constructed. The first iteration game was built using Rule-Maker platform. The second iteration, and the final game used in Phase 2, was built in Pathways platform. The full arrow of Diagram 3.1 illustrates that x is programmed in y, for example the Rule Maker was programmed in Imagine software. The dotted arrow in the diagram illustrates that x was a pre-version of y, so Rule Maker was a pre-version of Pathways. I now describe each of these elements in turn. I will elaborate the Imagine software and its two platforms below.

3.3.1 Imagine

Imagine is an object-oriented language and it was a recently devised version of Logo, developed in the Slovak Republic (see Kalas and Blaho, 2002). Dr. Ivan Kalas was also one of the main contributors at evaluation through discussion of the games (see section 4.2.2). In this study Imagine was invisible to the children and to me, as a designer of the game. Thus, I will not discuss its features further here. The reader may safely regard Imagine as opaque, much like any other programming language. For further details see http:// www.mathsnet.net/logo/imagine/.

3.3.2 Rule Maker

Rule Maker was a preliminary version of what subsequently became Pathways (see section 3.3.3). The reader will not need to know any details, but at this point it is important to outline the general idea of this platform. Rule Maker was an iconic programming environment. Its pieces fitted together in a way to aim at simple rule expression. The rules were constructed in an iconic form by the use of 'robots'. These robots obeyed their rule when the game was switch on. Figure 3.1. shows a rule expressed in Rule Maker.

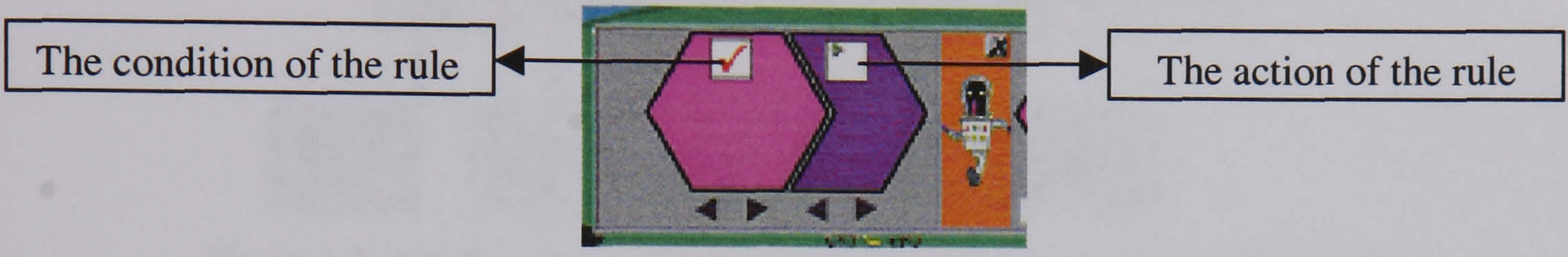

Figure 3.1: A rule expressed in Rule Maker saying ' I always move forward'. The robot in the rule will always move forward. Its condition shows the symbol of 'always' in Rule Maker and its action shows the direction of its movement.

Figure 3.1. shows that a rule in Rule Maker was constructed by a hexagon, which showed the condition of the rule, and by an arrow, which showed the action. The reader can refer to Chapter 5 (section 5.2) for further details, where the game that is used in the first iteration by using the Rule-Maker software is described.

3.3.3 Pathways

Pathways, like Rule Maker, was also a programming environment developed in Imagine. Pathways also allowed children to build and modify rules via a graphical iconic interface. A major period of my study was dedicated to designing Rule Maker and Pathways environments in Playground Project. Mainly, I was a beta tester in the design of Pathways. My role was to try out versions with children, making recommendations, and attending planning meetings with members of the team in order to improve the software for maximum expressive power for mathematical concepts.

Since the game that finally evolved for use in the Phase 2 study employed Pathways as a platform, I will now describe Pathways in some detail. I will explain the key features of Pathways as they appear on the screen when the user is introduced to the software. Figure 3.2 illustrates the main features of Pathways.

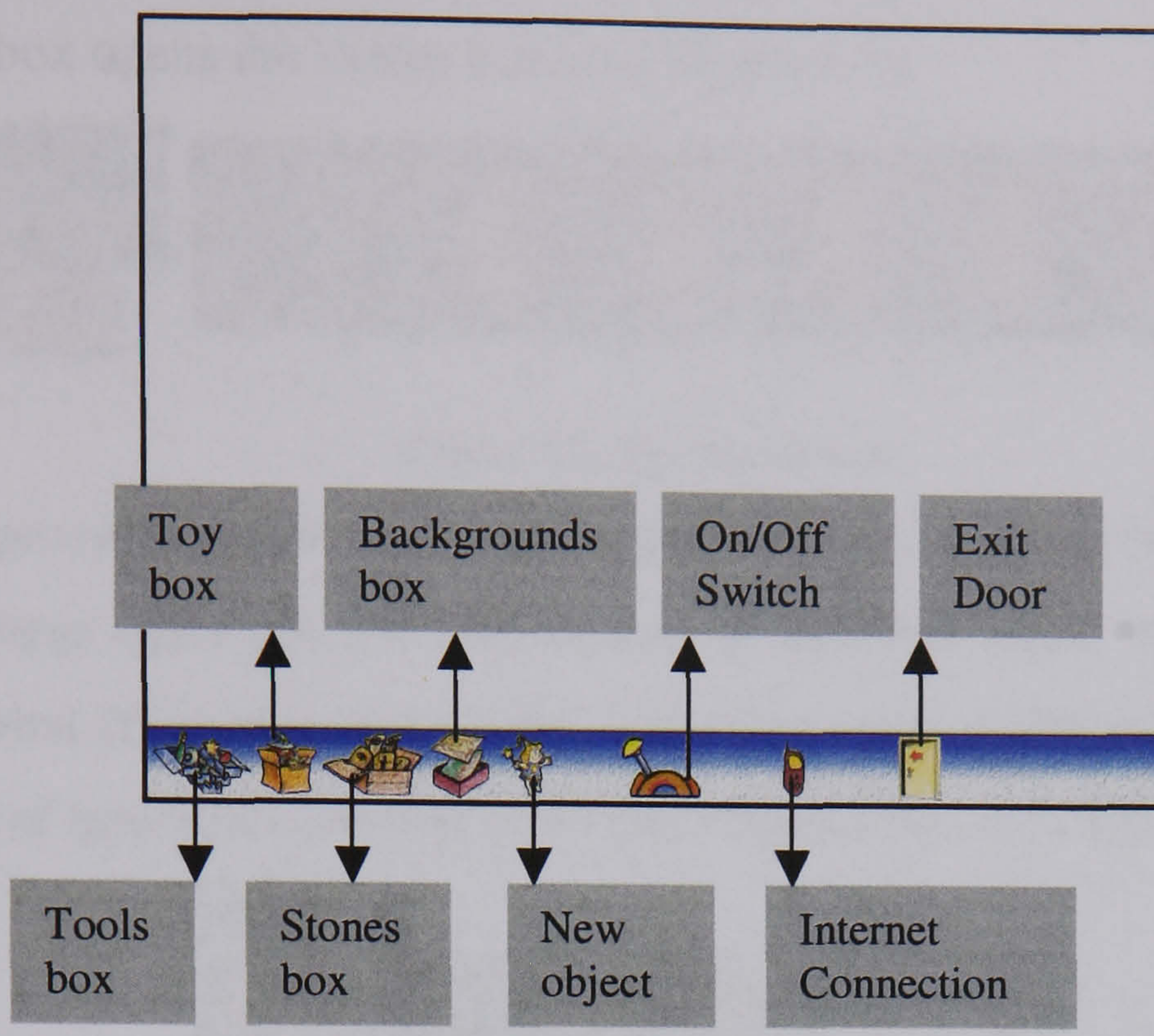

Figure 3.2: A Pathways screenshot, illustrating the main features of the system

On the bottom of the screen of Pathways software (seen in Figure 3.2) there are the Pathway's features. Reading left to right, the key features of Pathways are: 1. the tools box, 2. the toy box, 3. the stones box, 4. the backgrounds box, 5. the new object, 6. the on/off switch, 7. the internet connection, 8. the exit door. Selection of each icon , by double clicking, makes other icons appear on the screen as described below in more detail. The first icon opens the tools box (see Figure 3.3).

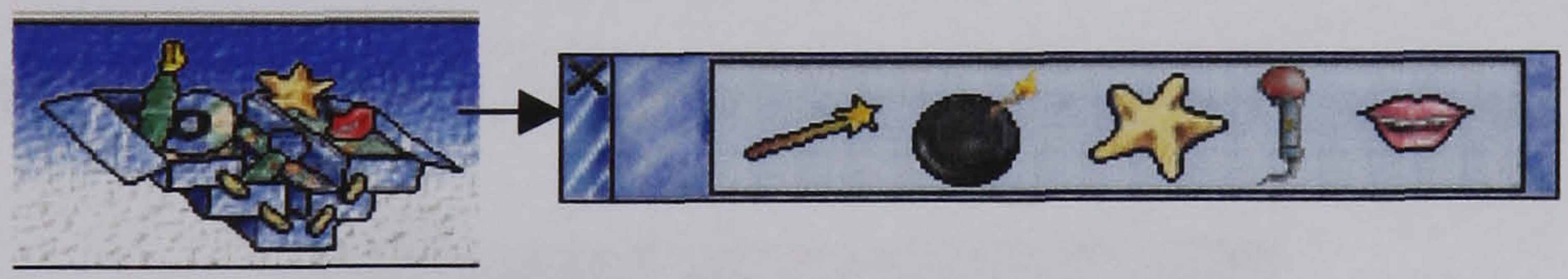

Figure 3.3: The tools box

By using the tools box (as shown in Figure 3.3) the children have the choice to copy objects on the screen by using the magic wand, to destroy objects by using the bomb, and they can change the shape, the speed and the heading of objects by using the star. They can also use the microphone to add new words in their objects and the mouth to listen to what each object is doing.

Near the tools box on the bottom of the Pathways screen is the toy box (see Figure 3.4).

Figure 3.4: The toy box

In the toy box children can find pre-constructed games or save their own. By selecting the icon next to toy box opens the stones box (see Figure 3.5).

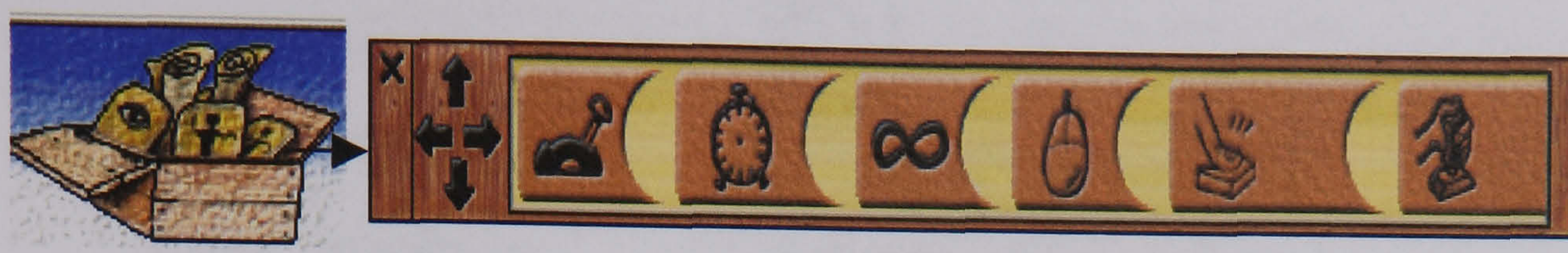

Figure 3.5: The stones box

The stones box provides children with the opportunity to construct rules beside each object in the game. These rules are the key-aspects of the Pathways, which will be of great importance in what follows in the study. I therefore, give a few examples to provide the reader a flavour of how rules are expressed (see Figures 3.6, 3.7, 3.8).

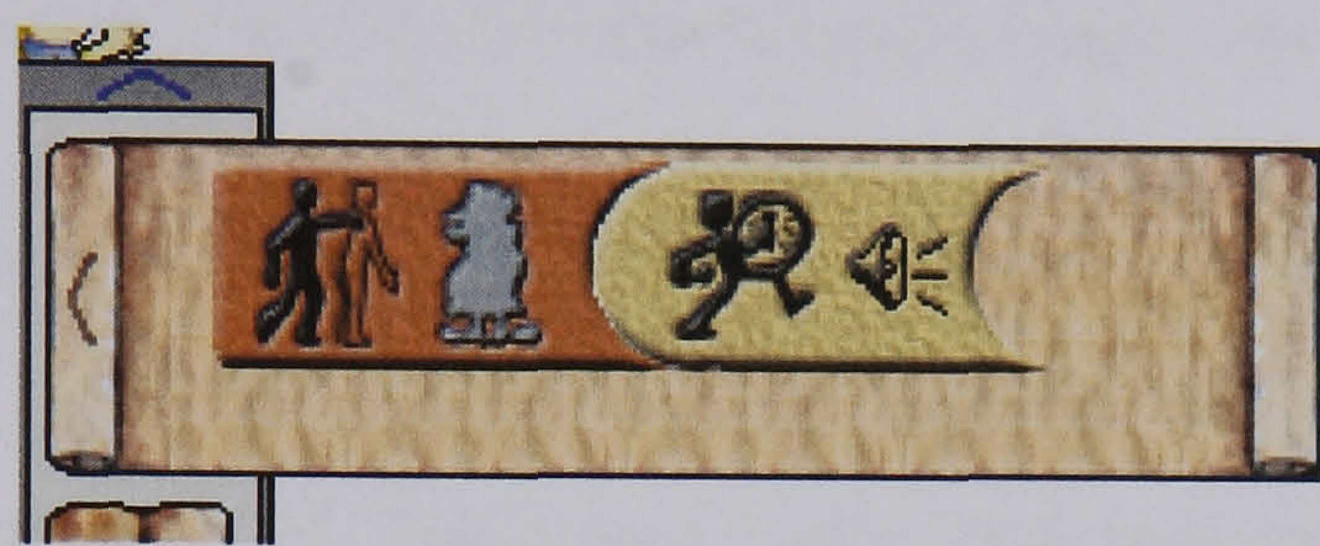

Figure 3.6: A rule in Pathways stating 'when I am touching any object I play a sound'

For example, in Figure 3.6 there is a rule stating 'when I am touching any object I play a sound'. The dark stone shows the condition of the rule and the light stone shows the action of the rule. When the children in Pathways pick up the mouth from the tools box they can hear what the rule is. The children can also add new stones to the rules and they can create more than one rule in any object of their game (see Figure 3.7).

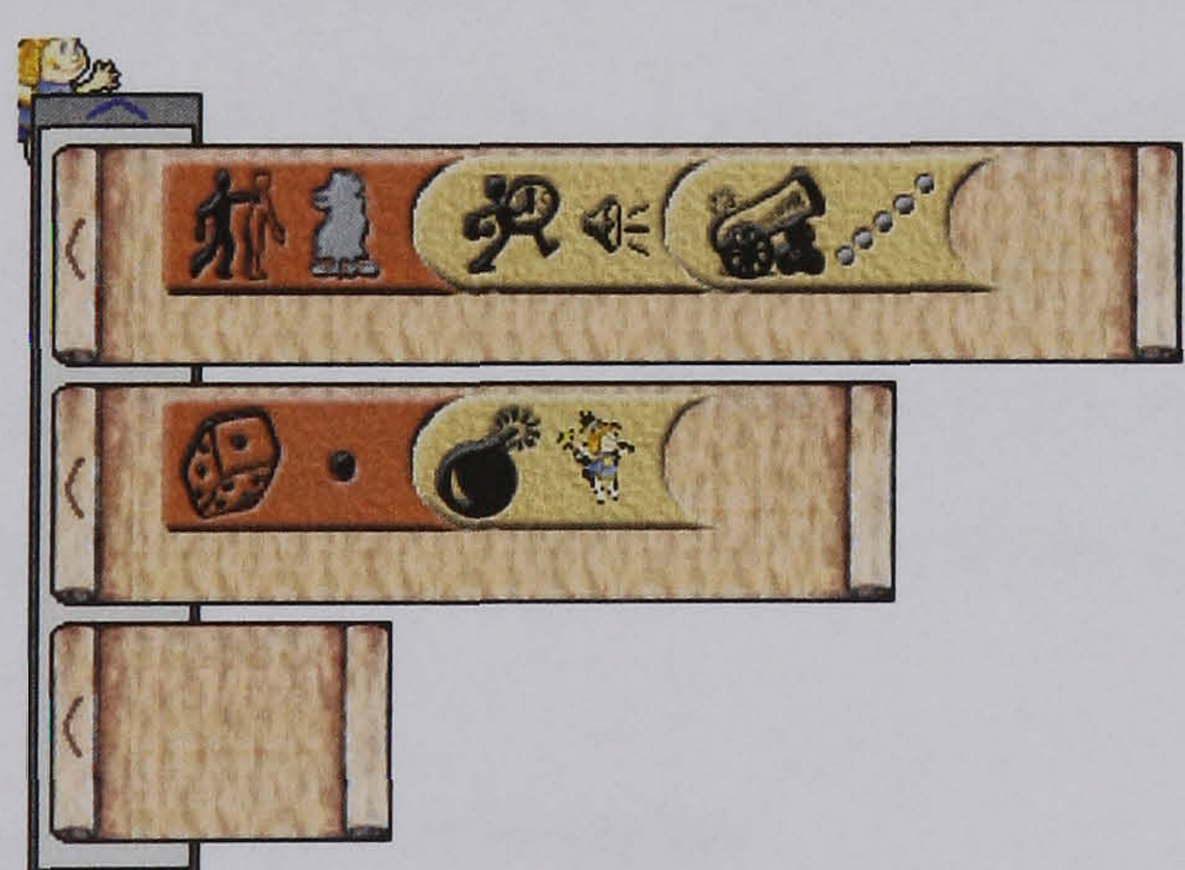

Figure 3.7: One object's rules in Pathways stating 'when I am touching any object I play a sound and I shoot a bullet' and 'when the dice lands on one I blow up myself'.

For example, in Figure 3.7 the first rule states 'when I am touching any object I play a sound and I shoot a bullet'. It is followed by the rule 'when the dice lands on one I blow up myself'. When the children construct the rules in an object the last rule is always empty in order to provide space for creating a new rule. By constructing rules at Pathways the user can also connect objects, with the message-passing feature (see Figure 3.8).

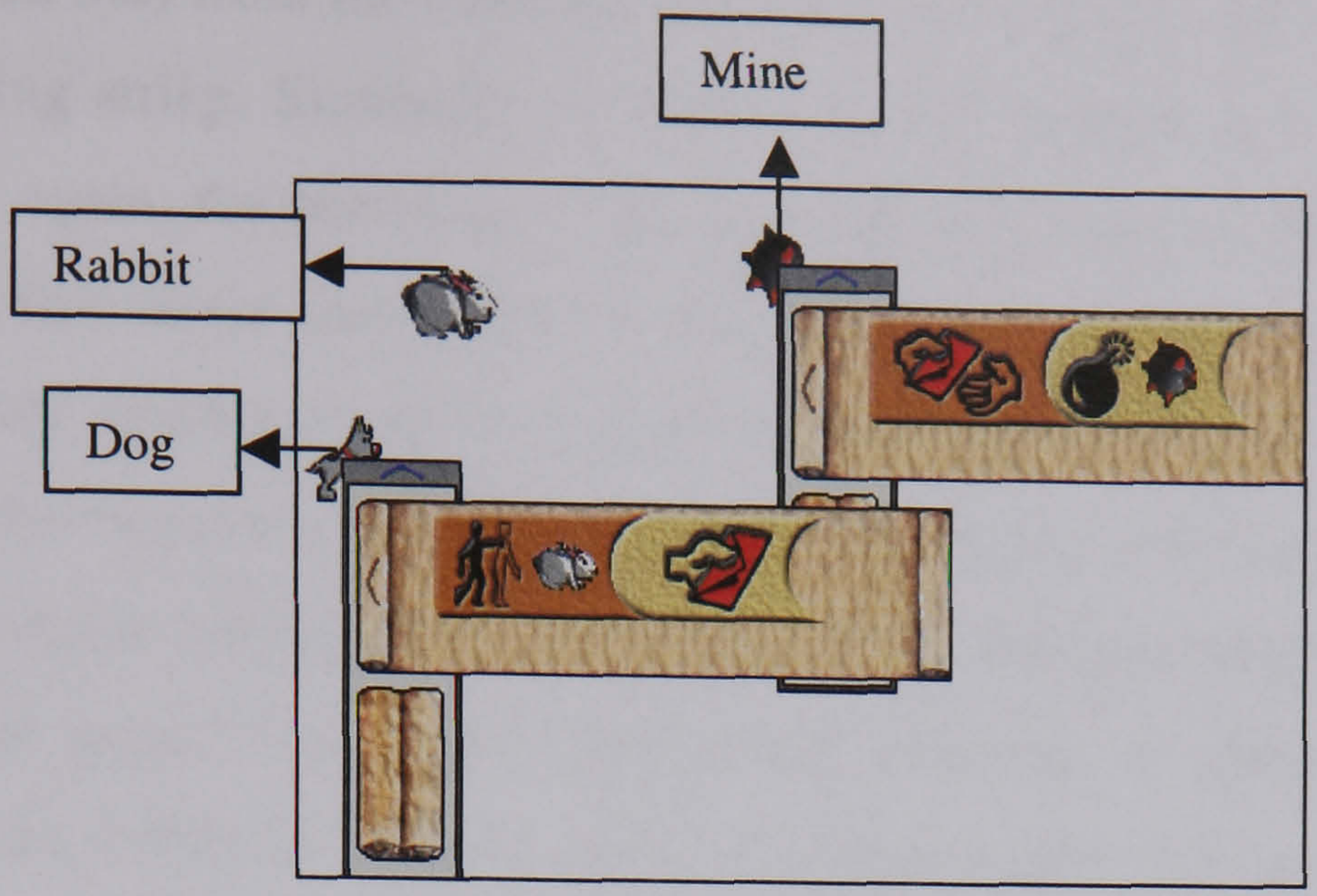

Figure 3.8: A passing message rule. The dog's rule states 'when I touch the rabbit I show a red message' and the mine's rule states 'when I receive a red message I blow up myself'.

Figure 3.8 shows a rule in which the dog says 'when I touch the rabbit I send a red message' and there is a mine that says 'when I receive a red message I blow up myself'. This message-passing feature of Pathways gives the opportunity to link the rules of different objects. This is important, since it relates an object's action behaviour with another object's condition.

3.4 The design principles of the game

The game took several forms. The details of these forms and the final version are given in Game Evolution, Chapter 5. Here, however, I will outline the principles of the design that informed the construction process and how they are related to the tool platforms, Rule Maker (Iteration 1) and Pathways (Iteration 2 and final game). Again, I will stress the part that concerns Pathways as this was the platform of the final version of the game.

3.4.1 The major design principles of the game

There were three major principles that governed the design of the game. These principles were:

1. *The manipulable sample space (and distribution).* A 'lottery machine' represented an "executable sample space" in the game[2]. The 'lottery machine' was a visible manipulable engine for the generation of random events and with it the children could directly manipulate the outcome of the game. For example, in iteration 1 (c.f.

[2] This is the reason that the game, from now on, will be also called as 'a lottery game'.

section 5.2) from the children's point of view, the shuffling of the balls caused the lighting strike. Similarly, in iteration 2 (c.f. section 5.3) from children's point of view again, the touching of the coloured balls caused the movement of the space kid. The direct manipulation and linked connections provided by the software allowed children to set in motion the mechanism to trigger an event, and be able to link the execution of that event with an outcome on the screen.

2. *The spatial representation of sample space.* The presentation of the lottery machine in the game was geometrical/spatial, whereas in previous work (for example Konold, 1989) the sample space was either hidden (i.e. not available for inspection or manipulation) or represented only in quantitative form (by using only numerical quantities). The lottery machine contained balls of different colours, which made it possible for children to carry out as many events as they like without being obliged to think about numbers (as they would have to using dice, coins, etc.). Moreover, in the final iteration, the children had also the opportunity to change the probability of an event to occur by changing the size of the balls and their arrangement.
3. *The existence of local and global events in the game and a visible link between them.* The lottery game gave the children the opportunity simultaneously to see on screen the local and global representation of an event of their sample space. A *local* event refers to the trial-by-trial variation and the *global* to the aggregate view of each single trial[3]. In practice, local events might be used by children to make sense of short-term behaviour of random phenomena, while global events are associated with long-term behaviour of a lottery game. Thus, whilst individual outcome could be seen as a single trial in a stochastic experiment, the totality of these outcomes gave an aggregated view of the long-term probability of the total events.

3.4.2 The 'concept' of a lottery game

As described in the previous section, the lottery game gave the children the opportunity, by manipulating the sample space and distribution, to identify intuitively whether an event is possible (whether it is impossible, certain or somewhere in between).

The lottery game consisted of two key pieces: a 'lottery machine' and a link between the local and global events. Diagram 3.2 gives the concept of the game.

[3] Many studies (for example Pratt, 2000 Ben-Zvi and Arcavi, 2001; Konold and Pollatsek, 2002; Rubin, 2002) have shown the importance of linking local understandings with the global ones, the aggregate view and how this can be a complex process for students.

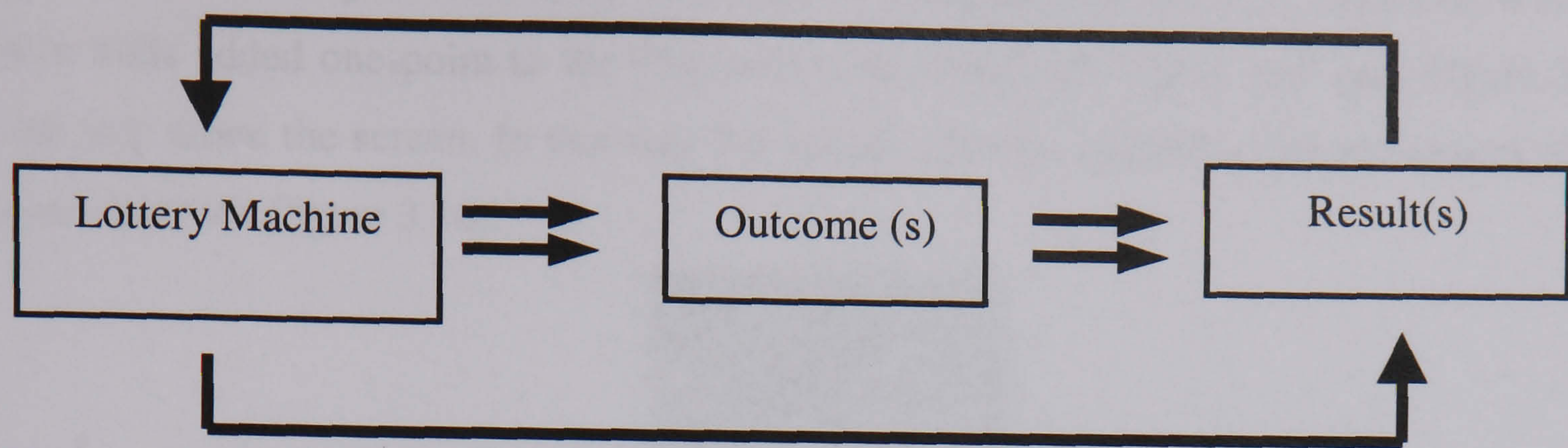

Diagram 3.2: The concept of the lottery game: the connection between the lottery machine, the outcomes and the results

Diagram 3.2 shows how the ingredients of the lottery game are connected. The lottery machine generated an outcome and this affected the result of the game. The short arrows illustrate how children, by manipulating the lottery machine, were intended to experience the outcome of an individual event in the machine (i.e. a collision between two balls) and how this was connected to a single result in the game (i.e. effecting a movement of an object in the game). The single outcome from the lottery machine provides an idea of a local event. The totality of the outcomes of the game gave a more aggregated view of the results and the lottery machine's construction. The manipulations made via the lottery machine could have a short-term and long-term outcome within the game. For example, children could make decisions about their next change in the lottery machine based on long-term result of their previous constructions.

3.4.3 A description of the different parts that comprise a lottery game

I will now illustrate each part of the lottery game in more detail. An example of a lottery machine can be seen in Figure 3.9.

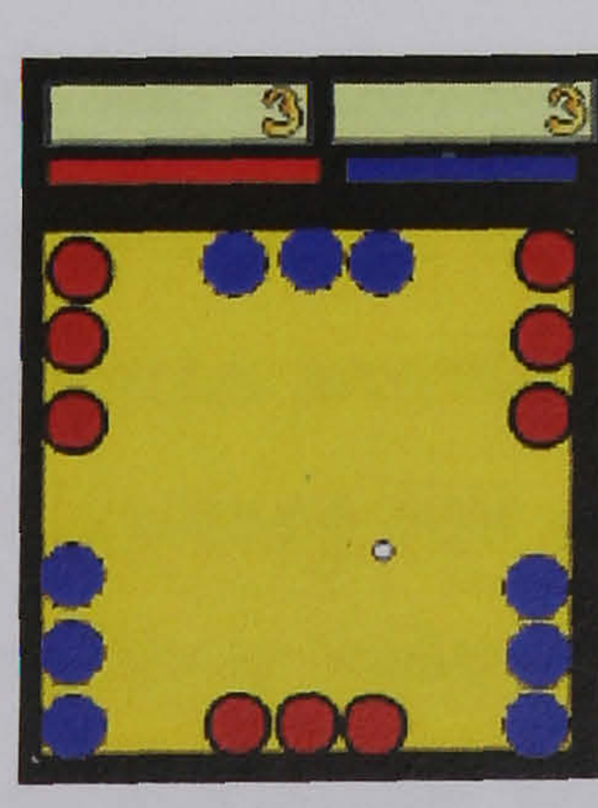

Figure 3.9: A lottery machine with its scorers showing the local events of the game

The lottery machine here is represented by the yellow square. In it, a small white ball bounced and collided continually with a set of static blue and red balls. Children could change and manipulate a number of aspects in order to construct their own sample space: the number, the size, and the position of the balls in the lottery machine, and they could also create new objects with their own rules. As programmed initially, collisions with the blue balls added one point to the blue score and moved the 'space kid' (see Figure 3.10) one step down the screen. In this way the lottery machine controlled the movement of the space kid (see Figure 3.10).

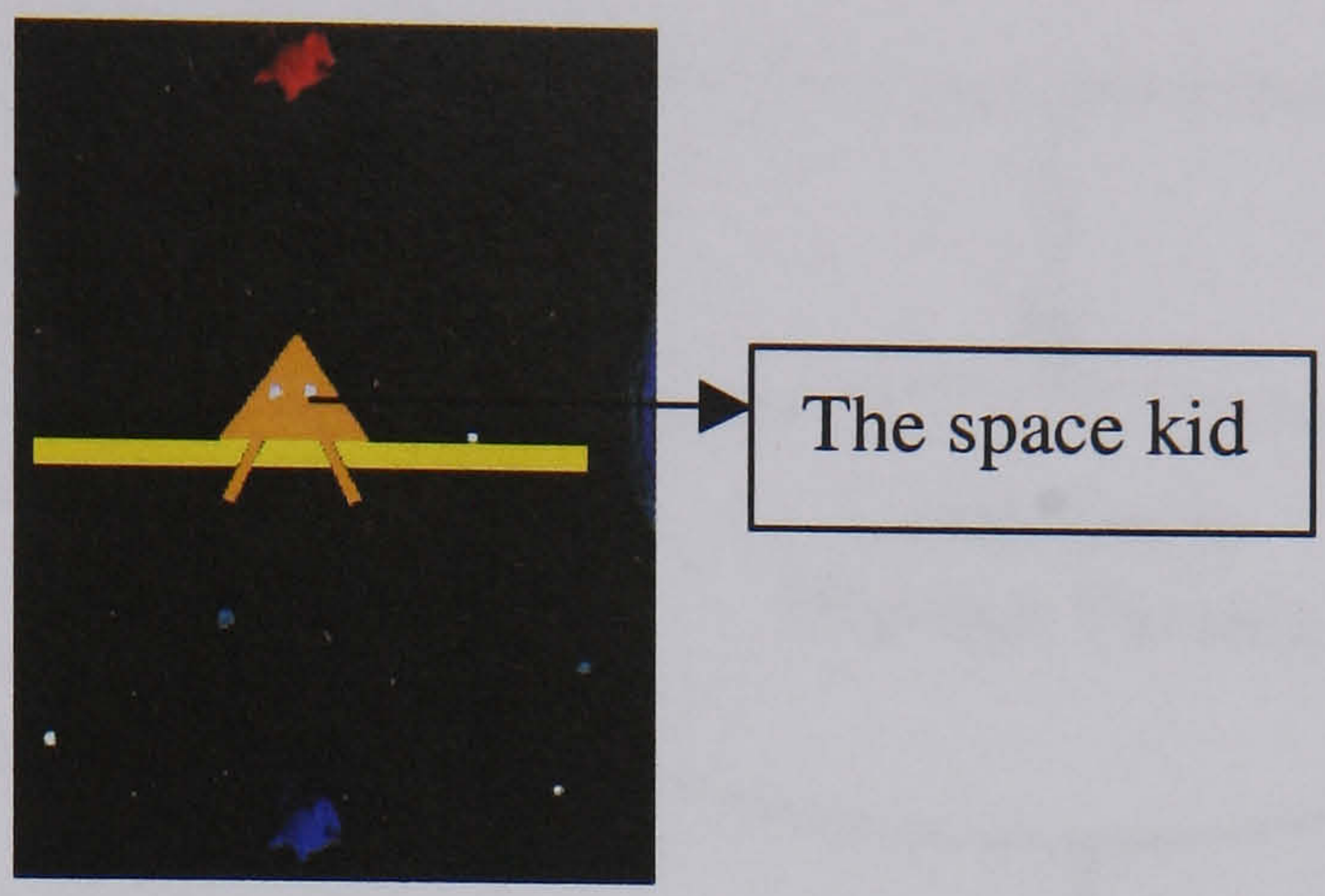

Figure 3.10: The space kid and the planets that represent the result of the game

Broadcasting and receiving messages (see section 3.3.3) achieved control of movement of the space kid. The continuous movement in 2-dimensions of the small ball in the lottery machine was designed in order that children might visualise the global outcomes of the game.

3.4.4 The choice of Pathways (Rule-Maker) for designing the lottery game

Pathways, like its predecessor Rule Maker, was designed as a medium where children can build and modify games using the formalisation of rules as tools in a constructive process. This enabled the construction of a lottery game, which afforded a simple means for programming the direct manipulation of objects, with which children could express meanings from actions and build new meanings of probabilistic ideas.

Hence, there were three reasons for the choice of Pathways (Rule-Maker): 1. evident, iconic rules that could be understood by children of this age, 2. easily manipulated objects and 3. a clear message-passing mechanism providing a mean of linking local and global events. Thus, rules could be created specifying how each object of the lottery game works. This afforded the children the opportunity to understand how the objects of the game were interrelated. It also allowed them to manipulate and link local and global events (c.f. section 3.4.1). Diagram 3.3 shows how the mathematical ideas and programming criteria are interrelated.

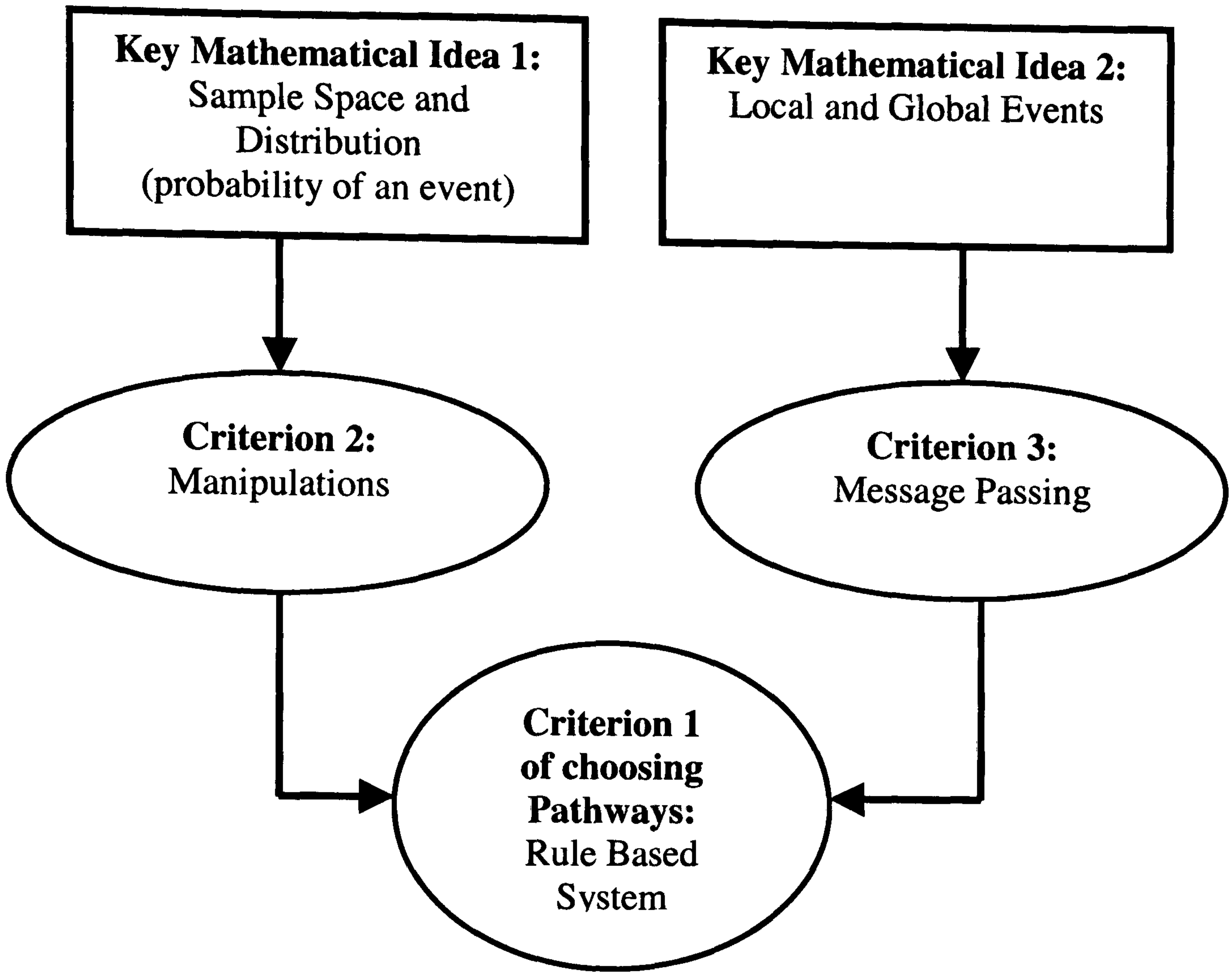

Diagram 3.3: The interrelations between the mathematical ideas developed in the game and the criteria for Pathways choice

Diagram 3.3 shows the two key mathematical ideas developed that formulated the design of the game. The key contribution of Pathways was that it allowed manipulations of the sample space and distribution, while its message-passing gave to the game the opportunity to link its local and global events. Both criteria were related to the rule-based system on which Pathways (and therefore the lottery game) was built.

3.5 Restating the aims

The study is focused on children's expressions of randomness. The aims of the study can be restated as follows:

1. an aim to document and discuss the evolution of the lottery game in 'Pathways' ('Rule-Maker'), and the interaction between its design principles and its realisation in the lottery game and,
2. to answer the question: when young children interact with the lottery game, what expressions of their informal intuitions of randomness are observed and how did the lottery game mediate the children's expression of chance events?

The procedure of reaching the aims was organised in two main phases. The first phase refers to the design and evaluation of the lottery game where the iterative design process takes place. The second phase refers to the learning investigation phase that concerns the second aim of the research, where the main study takes place. The following chapter describes in general the qualitative methodology employed for both phases. It subsequently provides evidence of how the data has been collected and analysed in each phase of the procedure.

CHAPTER FOUR

Methodology

4.1 Overview

Two main sections comprise this present chapter: the methodology of the iterative design phase and the methodology of the learning investigation phase. The first section describes the iterations within the iterative design phase, the analysis of iterative design phase, the qualitative approach for the evaluation through game use and the data of the task-based interviews. The second section refers to the methodology of the learning investigation phase. This section describes the data collection and data analysis of the final iteration and it refers to the tilt box experiment and the children's experience with Pathways.

4.2 Phase 1: Iterative Design Phase

As mentioned in Chapter Three, the research was developed in two phases: the iterative design phase and the learning investigation phase, corresponding to the two aims of the study. This section deals with Phase 1. *Iterative design* was defined by diSessa (1989) as follows:

> 'one must carefully observe and document the activities of children in prototypes of the proposed microworlds...including some sense of the span of conceptual states that children might be in.' (p.216-217).

Iterative design proposes that by using the game[4] with learners the researcher tries to obtain feedback and insights that will feed into the next iteration of game development and use. The method that I followed in designing the lottery game was that children were presented with a set of problematic situations, which sought to develop their probabilistic knowledge in the context of the game; at the same time this gave feedback and insights about the game that fed into the next iteration of game development. The iterative process facilitated the gradual refinement of the game and a gradual focusing of the primary design issues. Bearing in mind that Pathways was developed as the study developed, the iterations

[4] I remind the reader that the 'game' is used in a special way here, as described in Chapter Three.

also gave feedback to the Pathways software itself, by suggesting new functions for the next version of the software that could be used to develop the game. An example of giving feedback to the game's software was when a 'stone' (see section 3.3.3) needed to be created first in Pathways in order to be used in the following version of the game. The researcher had to describe the functions of the new stone and how this would be used in the game. Specifically, a 'bouncing stone' was needed to develop from iteration 1 to iteration 2 and this stone had first to be created in Pathways, and then used in re-designing the game for iteration 2. Diagram 4.1 illustrates the process of iterative design.

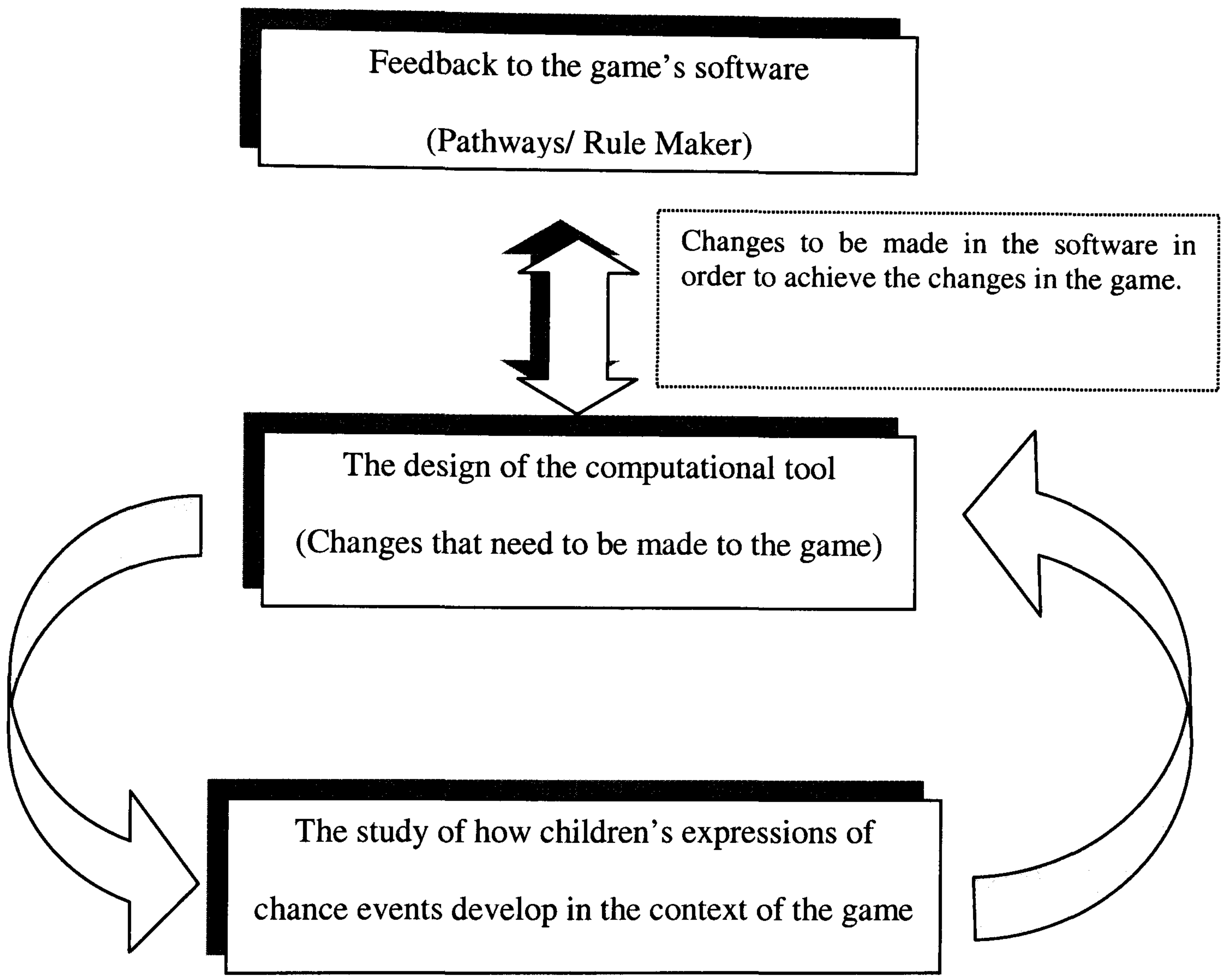

Diagram 4.1: The iterative design of the computer game

Diagram 4.1 highlights the two main issues that concern the study: the design of the computational tool and the children's expression of chance events in the context of the game. These two issues are represented in the two big rectangles. The arrows between them show the iterative design process that takes place. The interaction between the design of the game and the Pathways software is also shown. This illustrates that while the Pathway's structural features shaped the ways in which the design and implementation of the game evolved, the game's priorities and preliminary testing also fed back to shape the evolution of Pathways.

When the game, after a number of iterations, broadly satisfied the first aim of the study, then the iterative design stopped and the learning investigation phase (Phase 2) started where the main data collection was carried out. A difficulty in iterative design was generally to decide when the cycles of design came to an end. In my study the process came to an end when the game seemed to be suitable enough to be used for the learning investigation phase and ready to reach the second aim of the study.

In the iterative design phase each iteration consisted of two phases, game *development* and game *use*. The learning investigation phase could be also considered as a final iteration of the study. Thus, its focus represented a decisive shift from game development to game use.

Game development and game use are described as follows: the first gave feedback on the current design of Pathways and it fed into the development phase of an iteration and, where relevant, (see iteration 2 in Chapter Five) the development of new games. During an iteration the 'game use' phase generated and analysed the choices children made, how these choices were influenced by the structure of the game, the ways in which it was used and the connection between the use of the game and the expression of children's probabilistic thinking. Diagram 4.2 gives a linear representation through time of the iterations of the study.

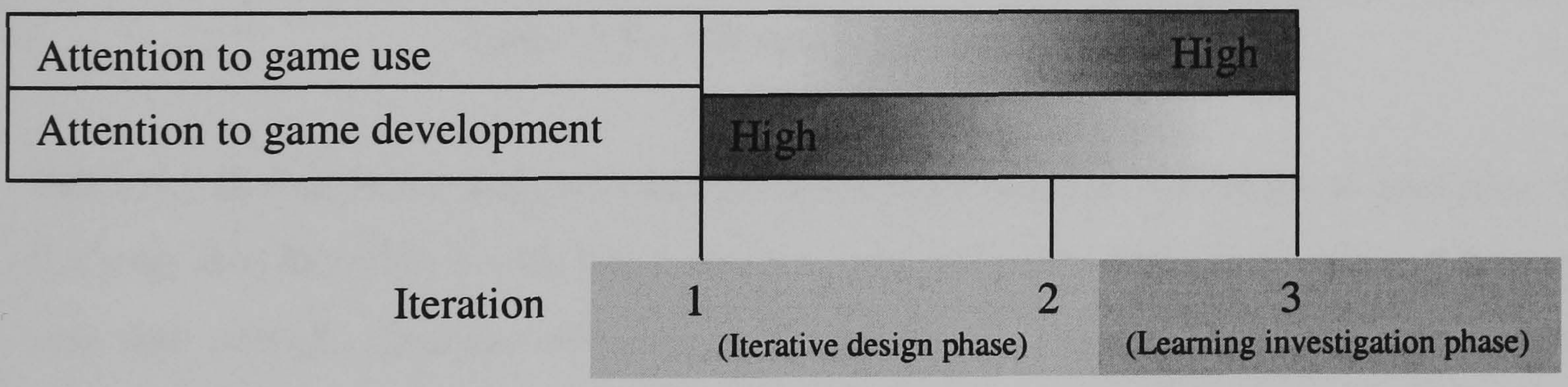

Diagram 4.2: Linear representation through time of the iterations of the study

Diagram 4.2 shows that the study developed through three iterations that switch attention from game development to game use. Firstly, the game development was based on initial interpretations of the literature (see section 2.3.3.3) and personal experience (my experience as an educator interacting with children of this age) and became more sophisticated after children interacted with it[5].

[5] The changes of the game based on children's interaction are presented in Chapter 5.

4.2.1 Iterations within the iterative design phase

As indicated in Diagram 4.2 the first two iterations refer mainly to the development of the lottery game rather than to its mediation of children's expressions. The iterative design phase was conducted through semi-structured interviews. The methodological details are discussed later, but the overall schedule is set out in the Table 4.1.

Iteration	1	2
Date	March 2000	April 2001
Number of children	2 pairs of children (3 girls and 1 boy)	1 pair of children (1 boy and 1 girl)
Age of children	5:6, 6:0, 7:0 and 7:6 years old	Both 7:0 years old
Background of children	mid-socio-economic mid range of mathematical ability	mid-socio-economic mid range of mathematical ability
Duration	2 x (1.5 to 2 hours) each pair 1x1 hour all four children together for discussion	1x (1.5 to 2 hours)
Place of interview[6]	Playground Project Lab Institute of Education, U.K.	Computer Lab All Souls School, U.K.
Children's experience with the software[7]	6 months	1 year and 6 months

Table 4.1: The schedule for the iterative design phase

Table 4.1 describes the date, the number of children, duration and place of interview. Thus, it shows that Iteration 2 took place about a year later than Iteration 1. One of the reasons was that several attempts were made to satisfy the changes that Iteration 1 dictated. Another reason was that in order to make the changes to the game there had to be changes in the game's platform (see diagram 3.1). So, after eleven months of iterative design of the game's software it emerged that it was necessary to create a new version of software, called instead of 'Rule-Maker', 'Pathways'(see section 3.3), and the iteration 2 experiments took place in April 2001.

Another six children were planned to participate in Iteration 2, but some immediate changes were indicated after the first interview. Thus, the other interviews were deferred to

[6] In the place of interview were only the children and the researcher.
[7] The children were participating in the Playground Project.

iteration 3, the learning investigation phase. The game use and game development took place as the interview was taking shape.

The interview at the All Souls School took place in the afternoon after children had finished their lessons. All the interviews were made with the cooperation of children's teachers and parents[8]. The children were characterised by their teachers as average mathematical ability. The children who participated in these two iterations were part also of the Playground Project club and they had previous experience on working with Rule Maker software (iteration 1), and Pathways software, as was named after iteration 1.

4.2.2 The analysis of the iterative design phase (Phase 1)

The analysis of the iterative design phase was based on six interacting processes: 1. epistemological analysis of the mathematical knowledge, 2. virtual experiments: sketches of what sorts of games might address the strands of knowledge identified the epistemological analysis, 3. design prototypes: sketches and drawings written down of what was required for the games, 4. constructing prototypes: the construction of the actual games developed in Pathways (Rule Maker) software and based on the design prototypes, 5. evaluation through discussion: the main contributors were Professor Richard Noss (Playground Project co-director), Dr. Dave Pratt (Playground Project partner) and Dr. Ivan Kalas (Playground Project partner), 6. evaluation through game use (see Chapter Five): the prototypes were tested and debugged through use with children. The interaction between these six processes can be seen as a hexagon (see Diagram 4.3).

[8] In working with children some ethical issues were taken into consideration. The age of the children (5½ -8 years old) made it necessary to get permission from their parents directly, or through their teachers, to work with them and make recordings. The children were also asked if they wanted to participate. The date and time of an interview, if it was at school, was chosen between the researcher and the teacher of the particular children. Also, the children were allowed to terminate the interview any time they liked.

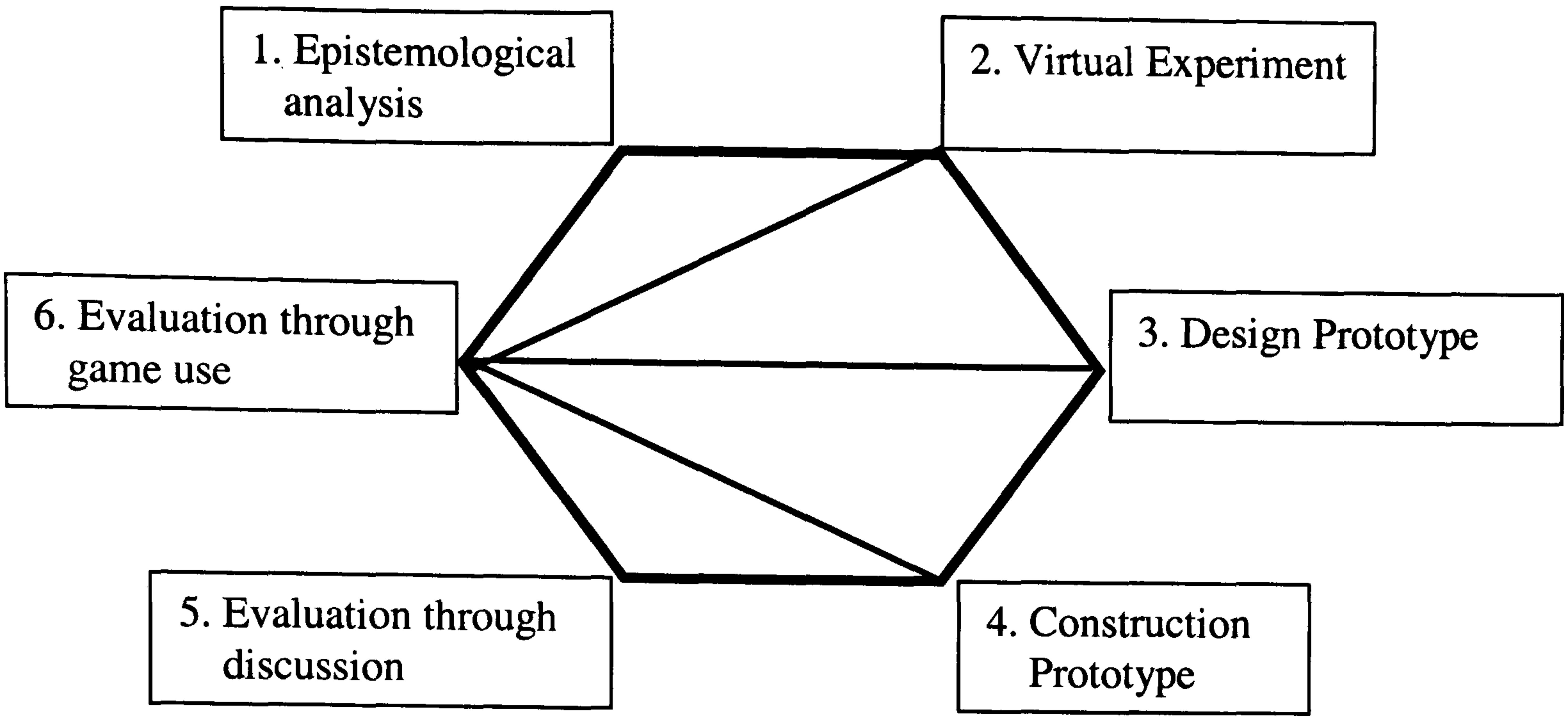

Diagram 4.3: The six interacting processes of iterative design phase

The elements of Diagram 4.3 were developed and analysed. Although the sequence of the processes of the iterative design phase was linear, the evaluation through game use process can be seen as a long-term data analysis and it was also assessing the whole iterative design phase. The evaluation through game process gave also rise to a set of new design criteria to drive the next stage of iteration.

The evaluation through game use included the data collection from children task-based interviews. The game use evaluation is also the process of analysis that referred to the learning investigation phase of the study (Phase 2). The precise methodology employed for the evaluation through game use will be discussed in the following section.

4. 2. 3 The evaluation through game use

The complexity of understanding children's expressions and constructions suggested the use of a *qualitative approach* to achieve the aims of this study. This study exploited several forms of expressions: a. the button clicks, menu choices and various ways of pointing on the screen, the construction that they made in the lottery machine of the game, the use of the features of the Pathways software and the discussions between myself and the children, which were often used to validated and probe more deeply into the thinking behind their actions and discussions.

4.2. 3.1 Semi-structured task-based interviews phase 1

During each iteration the children were interviewed on their performance on a set of tasks. The tasks were prototyped with Professor Richard Noss and there was a list of issues to be addressed and questions to be answered. However, these interviews were flexible in terms of the order in which the topics were considered and allowed the children to develop ideas and expand more widely on the issues raised by the researcher[9].

The task-based interviews that were used can be called as 'semi-structured'. The *semi-structured* interview is one that has weak focus (a limited degree of control exerted by the interviewer over its contents) and a weak frame (a light control over the timing and duration of the interview) with opportunities afforded to the interviewee for review or editing and the construction of an informal setting (c.f. Scott and Usher, 1999; Powney and Watts, 1987). Robson (1993) proposes that in a semi-structured interview, the interviewer must work out a set of questions in advance, but is free to modify their order based on what seems most appropriate in the context of the interview, or change the way they were worded, give explanations, leave out some questions which seemed to be inappropriate with a particular interviewee or use additional ones. The interviews of the Phase 1 were based on a protocol of tasks that are described in detail in Chapter 5.

4.2.3.2 Equipment for data collection of the task-based interviews

In order to achieve faithful recording, the semi-structured task-based interviews of the Phase 1 were audio-taped and video-taped. The video camera was mainly focusing on the actions played out on the screen, it was also focused on the body language and the movement of hands, and the children's discussions and my interventions were overlaid onto an extra audio-recorder. The equipment[10] that has been used in my semi-structured interviews is shown in Figure 4.1.

[9] Piaget (1929) pioneered the clinical interview in the context of cognitive psychology. He states that:

> 'It is so hard not to talk too much when questioning a child, especially for a pedagogue...The good experimenter must in fact unite two often incompatible qualities; he must know how to observe, that is to say let the child talk freely, without ever checking or side tracking his utterance and at the same time he must be constantly alert for something definitive...' (p.8-10).

[10] Denscombe (1998) states that at one level, the audio tape-recorder and the video camera are reliable research instruments, as they capture the proceedings on the permanent record, and they provide an objective record of the proceedings in the sense that these research instruments do no interpret the events, they simply store them.

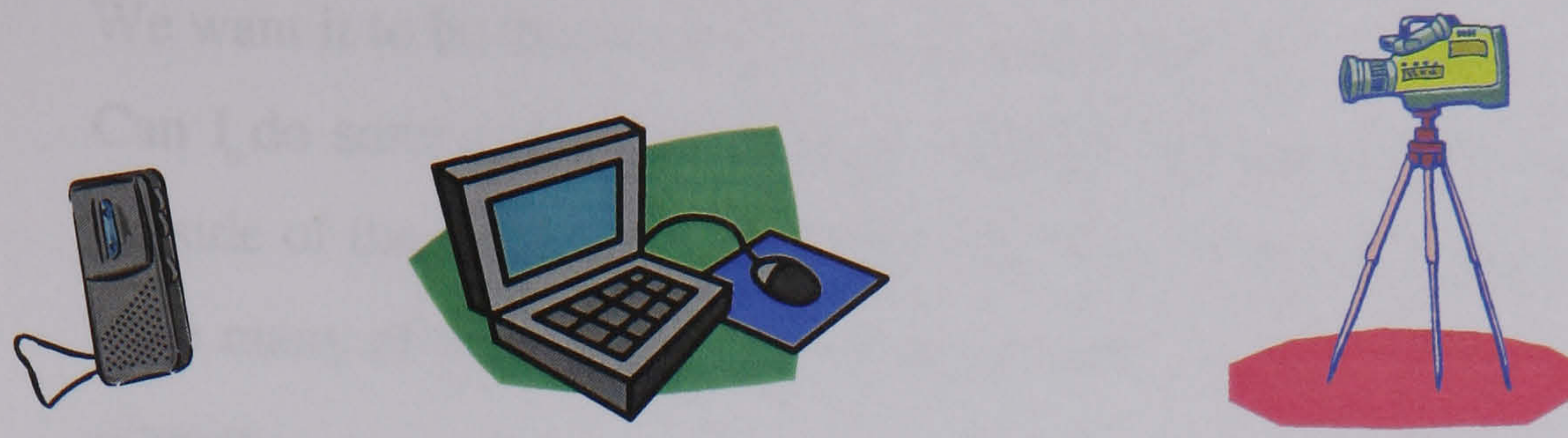

Figure 4.1: The equipment that has been used in the semi-structured task-based interviews

The validity and reliability of such approaches has been intensely contested. On the one hand, the direct contact at the point of the interview means that data can be checked for accuracy and relevance, as they are collected. On the other hand, the impact of the interviewer and of the context means that consistency and objectivity are hard to achieve. For this reason, the role of the researcher during the interviews needed to be defined.

4.2.3.3 The role of the researcher in the task-based interviews

The role of the researcher was that of a *participant observer.* The general aim of the interviews was for the children to make their own decisions in order to understand the aim of the task, and to construct their own 'lottery machine' (see section 3.4.1) based on these decisions. In the role of participant observer, there were times when it seemed appropriate to make different types of interventions[11]. The role of my interventions fell into the following categories: probing, explaining, experimental and technical (c.f. Pratt, 1998).

Probing interventions aimed to make children's thinking transparent when it came to inferring the reasons or intuitions that might lie behind their actions. Children were asked to give explanations of what they were doing by describing it to a peer, if they had one, or to the researcher, and to explain their reasons for doing it. An example of a probing intervention is the following where Lisa (L) and Sam (S) try to construct a sample space:

Lisa: Oh…no, no! Get this ball here and these on the side!
Researcher: Why?
L: If (it) gets out to go there…
R: So…

[11] Burgess (1984) claims that:

'Participant observers are involved in face-to face relationships with those who are researched, and that the observers are part of the context that is being observed. This results in the possibility of researchers modifying and influencing the research context as well as being influenced by it themselves.' (p.79).

This stance indicates that as well as observing though participating in activities, the observer can ask the subjects to explain various aspects of what is going on.

Sam: We want it to be blocked here. (He makes a circle.)

L: Can I do something? I'm going to explode all these balls (the others outside of the circle), because if we have less of these (balls), then we have many of these (the other coloured balls).

R: Hmmm...

L: I will explode them, because then the whites, we, will have less, or they are going to change the number over here (to the score).

My role in the above episode was to probe Lisa to describe what she was doing and why she was doing it. This made it possible for me to understand her thoughts and also gave to her partner the opportunity to follow her thinking and add his own ideas. For this reason, I used probes[12] to get the children to expand their thoughts. There were occasions during interview when the researcher wanted to delve deeper into a topic rather than let the discussion flow on the next point.

Explaining interventions were made when the children asked for explanation of a situation. Almost always, these questions were turned back to the children by asking what they thought, or by suggestions that they can try it out themselves (for example see Appendix 4, lines 232-233). Such an intervention was not entirely neutral since the children no doubt inferred that this was not a completely dead-end direction to follow, and so were perhaps unintentionally encouraged to follow that route.

Experimental interventions sought to make some change in the direction of the activity with possible implications for conceptual change. A purpose of these interventions was, for example, because the children were stuck (for example, see Appendix 4, lines 347-348). Sometimes this was obvious to the children and they sought advice as to how to proceed. On such occasions, I offered a suggestion which might lead to a new direction. Another purpose of using an experimental intervention was to explore whether a child was able to work with a new idea. Such interventions were only used when a child seemed to be particularly confident and already performing with some fluency.

Technical interventions were made to give explanations about the software as the clinical interview progresses, and sometimes or to overleap bugs in the software. These

[12] A probe is used in the sense of Robson (1993), who defines it as a device to get the interviewee to expand on a response when you intuit that she or he has more to give.

interventions mostly took place at the beginning of the interview when children were trying to understand the rules of each object in the game, becoming less frequent as the interview progressed. An example of a technical intervention is the following:

Sam: Well, I will make a new rule...

R: *How?*

S: I don't remember.

R: *Can we just continue with this rule?*

Lisa: Yeah...we have to find the sound here. (She finds the sound stone, she makes the rule and switches on the game again)...Yeah!

The suggestion of continuing with a rule instead of creating a new one made Lisa to remember the use of the sound stone from the stone library.

4.2.4 Data analysis of task-based interviews of Phase 1

The development of the lottery game took place through the iterative design phase during which the games themselves were modified, reflecting a new insight into the way that those games probed children's meanings for randomness and probability. The study of the children became increasingly systematic as the games converged on a design, which appeared to be effective in enabling the observation both of initial meanings for the stochastic, and the subtle changes in those meanings as new connections were forged during activity. In Phase 1, the episodes were delineated according to three broad codes:

A. *The design of the lottery game*: This concerned the changes that needed to be made in the lottery game in order to make it more interesting in the context and more transparent in developing and exploring children's probabilistic intuitions.

B. *The design of the software*: Bearing in mind that the software was developed as the study grew, each of the two iterations also gave feedback in the design of the software, by suggesting new functions to feed into a new version of the software. The new version worked for the task as the next iteration.

C. *The children's expressions of randomness*: The data of each iteration were analysed in order to find out how children made sense and use the ideas of randomness, fairness, distribution, certain, impossible events, probability of an event.

The above general codes for analysing the data were based on the previous literature review. The stress on this analysis was given to the design of the game. In the iterative design phase the purpose of analysis was mainly aimed on the design of the lottery game and the design of Pathways and less on the children's expressions of randomness, as the last axis was the main aim of the learning investigation phase. Table 4.2 describes the methods used for analysing the data of the iterative design phase.

Iteration	1 and 2
Methods used to analyse data	▪ The videotapes and audiotapes of four children, Nichol[13], Kate, Jerry and Ellis were partially transcribed (iteration 1). The videotapes and audiotapes of children, Lisa and Sam, were transcribed in detail (iteration 2). ▪ The transcriptions of the semi-structured interviews were used to develop tags that represented possible emerging issues and the transcripts were annotated using these tags. ▪ The main issues were developed and discussed with my supervisor Professor Richard Noss by considering these tagged transcripts alongside the children's semi-structured interviews.

Table 4.2: Data analysis in the iterative design phase of each iteration

As indicated in Table 4.2, the audiotapes and videotapes were transcribed. The first transcription of audiotapes and then the transcription of videotapes was a method of checking the data with other sources (c.f. Denscombe, 1998). The data were analysed based on these three broad codes (the design of the lottery game, the design of the software, the children expressions of randomness). The emphasis of the analysis was given to the first two codes (the design of the lottery game and the design of the software). The initial analysis of the third code was based on the following sub-codes: C1: Expressing Randomness, C2: Certain and Impossible Events, C3: Probability of an Event. Some samples were discussed with my supervisor to assess the validity and finally applied once more to the data. Each iteration is presented in more details in Chapter Five.

4.3 Phase 2: Learning Investigation Phase

This section describes the general outline of the final interviews. It describes first the idea of the tilt box, with which each interview began and it also refers to children's previous experience with the software. The outline of data collection where the idea of the semi-

[13] All names of the children are pseudonyms.

structured interviews, the role of the researcher as a participant observer, and the ethical issues taken into consideration are the same as these have been already described on the game use of Phase 1. Moreover, this section describes the coding that was undertaken for data analysis of the interviews.

4.3.1 The tilt box experiment

At the beginning of each interview of the final iteration, before children interacted with the game, a device was used to question children about random mixtures, similar to Piaget and Inhlelder's (1975) tilt box (see Diagram 4.4). It consisted of a rectangular tray, which could be able to swing up and down by means of fulcrum fixed to its base sides. Eight equally sized marbles of four different colours, two of each colour, were arranged in the tray, and also there was a divider that can be placed in the middle of one of the sides. The researcher balanced the tray, made an arrangement of the marbles, and then let the tray go. For example, a typical initial arrangement might be simply 'red-red-blue-blue-green-green-purple-purple'[14].

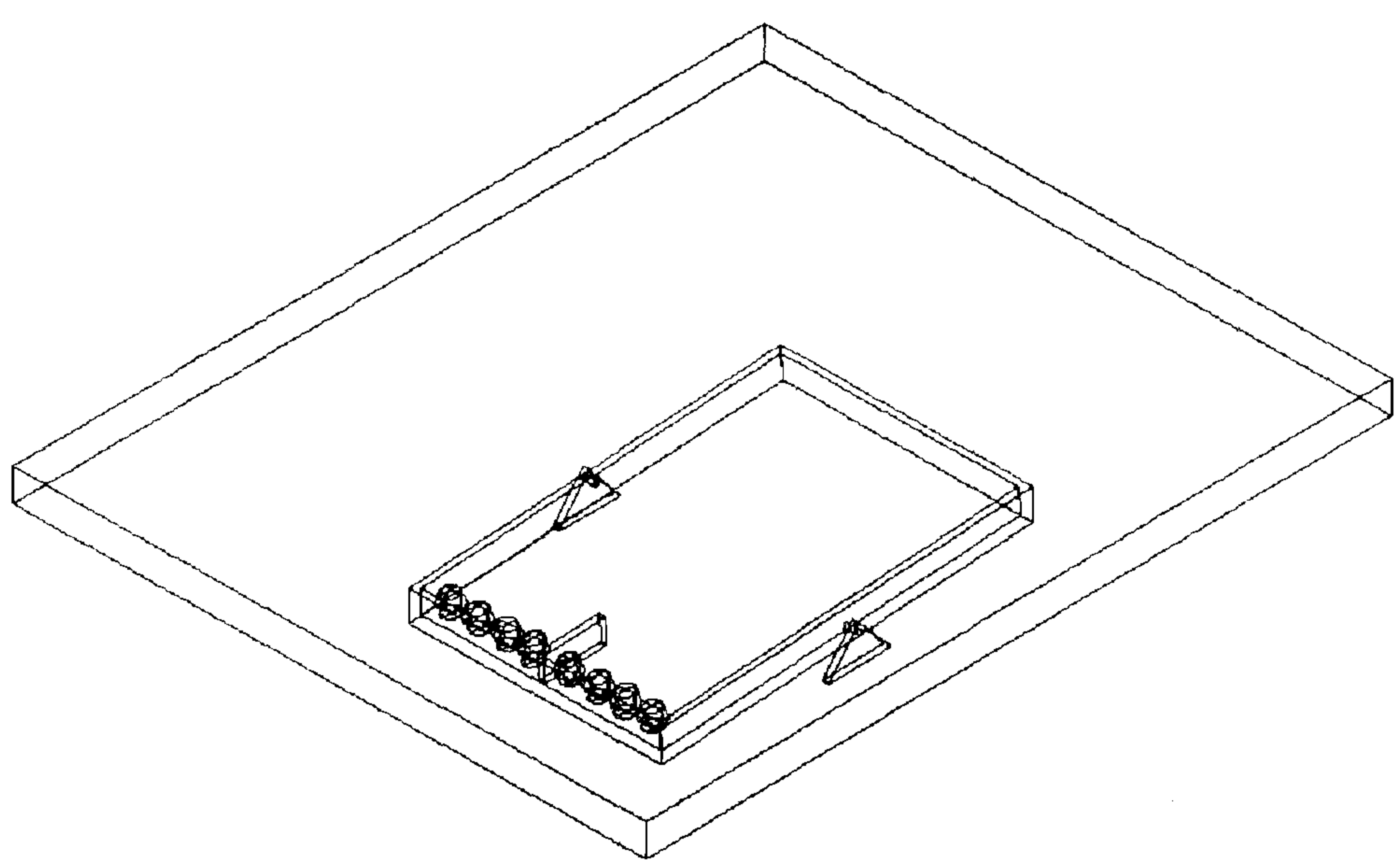

Diagram 4.4: The Tilt Box

Piaget and Inhelder (1975) state that children of the age considered in this research do not anticipate the 'irreversibility' involved in random mixture; instead, they tend to find any kind of order on the grounds of common properties of the elements or of their original arrangement. As a result, they are not able to start understanding what *chance* is.

[14] A snapshot of a tilt box experiment can be found in appendices (see Appendix A1).

The intention of undertaking this tilt box activity at the beginning of each interview was intended to investigate Piaget and Inhelder's claim. Another reason for doing their experiment at the beginning of each interview was to generate baseline data that would confirm whether the study was dealing with a representative sample of children, and also to assess the extent to which the *tool* mediated knowledge of chance.

4.3.2 Children's experience with Pathways in Phase 2

The data of this study was collected alongside the development of Pathways software (as described in Chapter Three). The children in this study were already experienced working with Pathways. The researcher trained them how to read and construct rules in the software and use the main parts of Pathways: 'tool box', 'stones', 'toy box', 'backgrounds'. They had already played games in Pathways (see Appendix A2) that introduced them to creating rules by using condition and action stones. They also performed tasks in Pathways that probed (a) children's thinking about conditions and actions and (b) children's understanding the meaning of showing and reacting to colours, sending and receiving messages (see Appendix A2.1). These activities seemed to be a necessary preparation for children to do the probabilistic task, in order for the software to be usable to the children and for the children's attention to focus on interacting with the task rather than understanding the software.

4.3.3 Data collection of learning investigation phase and associated methodological issues

The data collection of the learning investigation phase refers to the main collection of data. The methodological details are discussed later, but the overall schedule is set out in Table 4.3.

Phase 2: Learning Investigation Phase	
Date	September-December 2001
Number of children	23 children (12 girls and 11 boys)
Age of children	5:10, 6:0, 6:6, 6:7, 6:8, 6:10 6:11, 7:0, 7:3, 7:5, 7:6 years old
Background of children	mid-socio-economic mid range of mathematical ability
Duration	23 children x (2 to 3 hours each one) divided into four meetings, working individually[15]
Place of interview[16]	After school at researcher's house, U.K. After school at researcher's house, Cyprus
Children's experience with the Pathways software	23 children x1 hour each Working on tasks on Pathways. The context of the tasks is presented in Appendix A2.

Table 4.3: The schedule for the learning investigation phase

4.3.3.1 Semi-structured task-based interviews in Phase 2

The children were interviewed before (see tilt box experiment 4.3.1) and during their interaction with the computer game[17]. These interviews were videotaped and audio recorded and then transcribed in detail. The questions that were given in the interviews of the learning investigation phase were based on a protocol of questions and tasks (see Appendix A3). The aim of these questions was to prompt discussion and their nature changed through the iterations as certain issues become increasingly apparent, but the structure of each version of the interview was the same. The interview started with children trying to find the rules of the game and then to solve different problematic situations by making manipulations in the game, for example by manipulating the sample space.

The interviews were based on the 'Space kid' game, which is described in detail in section 5.4. The children in the final iteration worked individually. The interviews exploited discussions between the children and myself. Many times discussions were used to validate and probe more deeply the thinking behind their actions. These discussions gave a picture,

[15] The requirement of working individually was so that the children could not 'hide' some of their probabilistic ideas behind those of a partner, as this had happened in the previous iteration.
[16] In the place of interview were only the child and the researcher.
[17] The procedure was the same as in Phase 1 (see section 4.2.3.1).

not only of the performance, but also of the thinking that stimulated or arose out of these actions. Children were asked to try out their ideas, describe what happened, give their explanation of why this happened, and if it did not work, as they wanted, what else they could change. The activities of the task aimed at providing opportunities for the construction of meanings, affording a range of opportunities to use the game for the children to express themselves.

4.3.3.2 Equipment and the role of the researcher in the learning investigation phase

The semi-structured task-based interviews of the learning investigation phase were audiotaped and videotaped. The video camera was mainly focusing on the actions played out on the screen, it was also focused on the body language and the movement of hands, and the children's discussions and my interventions were covered onto an extra audio-recorder[18].

The role of the researcher was that of participant observer, interacting with the children in order to probe the reasons behind their answers and their actions. The general aim of the interviews was for the children to make their own decisions and manipulations in order to find a solution to the task's problematic situations. In the role of participant observer, there were times when it seemed appropriate to make different types of interventions either probing, explaining, experimental, or technical[19].

4.3.4 Data analysis of the learning investigation phase

The data analysis of the learning investigation phase was firstly based on the key mathematical ideas (sample space and distribution), which were represented by the problematic situations of fairness, unfairness and probability of the occurrence of an event. The data analysis of this phase was developed as Diagram 4.5 shows.

[18] The procedure was the same as in Phase 1. For more details see section 4.2.3.2.
[19] For more details the reader can refer to section 4.2.3.3.

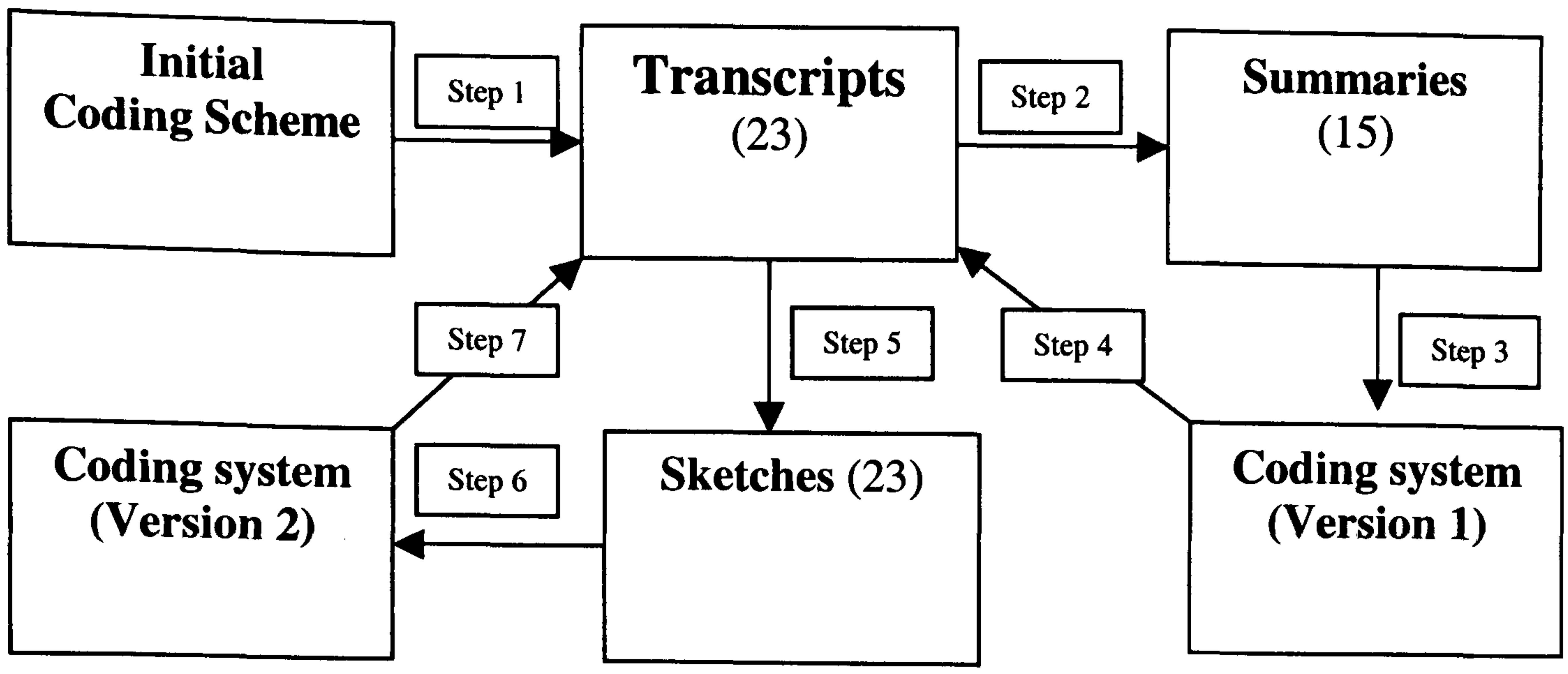

Diagram 4.5: The procedure that has been followed for the data analysis of the learning investigation phase

Diagram 4.5 describes the procedure that followed for the data analysis of the learning investigation phase. The *initial coding scheme* was based on the initial interpretations of the literature review (see section 2.4), my experience as an educator interacting with children of this age and on the aims of the study. The initial coding scheme of the study provided to some basic headings: Expressing Randomness, Certain and Impossible Events, Probability of an Event. These headings were used as a step 1 for an initial analysis of the *transcripts*[20]. The transcripts were presented in columns, giving also the line numbering of the transcript, the code for the content and the screen-shot of the computer game. Sometimes some notes could also be included in the transcript based on notes taken during the process of transcribing or annotations that gave a richer meaning to the words that were spoken. As some interviews were undertaken in the Greek language, the transcripts were translated into English by the researcher and well checked by a bilingual speaker of Greek and English. From the transcripts some *summaries* were created to describe how the thinking of the child of each transcript was expressed (see Appendix A5, for an example). The summaries of the transcripts were the step 2 of the data analysis and helped the researcher to develop a first more elaborated coding system of the transcripts (step 3). In the *first version of coding system*, some sub-headings were added to the initial coding scheme. For example, in the heading of expressions of random mixture were added the sub-headings: unsystematic movement, changing the mechanism. Using this coding system, the researcher went back to the transcripts and tried to code them (step 4). The coded transcripts (coding system version 1), gave the researcher the opportunity to make a *sketch of each child* (step 5, see Appendix A6, for an example) and thus, to create new version of coding system (step 6), validated by examples to researcher's supervisor. The

[20] A transcript can be found on Appendix A4.

transcripts were finally coded (see Appendix A4) based on the new version of the following coding system (step 7).

4.3.5 The final coding system of the transcripts

The final coding system of the transcripts was based on four general groups: a. local and global thinking, b. expressing the rules, c. engagement with the game and d. expressions of random mixture. These groups were divided into different categories as follows:

A. Local and Global Thinking

A1. Initial descriptions of the game

A1.1 Outcome oriented

A1.2 Sample space oriented

A2. The use of global evidence in judgements of equality and proportional thinking.

The codes here were used in order to find out how the child expressed the local and global events in the game (see section 3.4.1). The codes were also used to see how each local and global events in the game have been used by children to judge their thinking of equality and proportionality.

B. Expressing the rules

B1 Expressions for the rules of the objects

B1.1 The rules of the objects are expressed by using the third person.

B1.2 The rules of the objects are expressed by using the first person

B2 The children are looking at the rules to explain the unpredictable and to explain the movement of the ball.

B3 If the relative numerical strength is in terminal then looking at the rules is sufficient.

The codes of how children expressed their rules were looking at how each child expressed the rules of the game and how the child made use of the rules in order to explain or judge randomness.

C. Engagement with the game

C1 Use new objects

C2 Make changes to the game just to have fun.

D. Expressions of random mixture

D1 Unsystematic movement

D1.1 Liberal: e.g. 'Moves where it wants to go', 'By it self'.

D1.2 Controllable: e.g. 'The computer controls it'. 'It knows, we don't'.

D1.3 Control the uncontrolled situation.

D2 Changing the mechanism

D2.2Changing the variations of the white ball

D2.2.1 Change the number of the white balls.

D2.2.2 Change the starting place of the white ball.

D2.2.3 Change the speed.

D2.2.4 Change the size.

D2.2.5 An attempt to change the rules of the white ball / to control the movement.

D2.3 Change / Move the objects of the game

D2.3.1 Change the place of the two planets.

D2.3.2 Change the rules of the balls.

D2.4 Create new objects

D2.4.1 Having new objects in the sample space.

D3 Spatial Representation of sample space / Changing the variations of sample space

D3.1 Construction of fairness

D3.1.1 Moving and changing the number of the elements of sample space

D3.1.1.1Symmetrical teams

D3.1.1.2Making a pattern

D3.1.1.3Making circles

D3.1.2 Changing the size of the elements of sample space

D3.1.3 Changing the arrangement of the balls

D3.2 Construction of unfairness

D3.2.1 Certain and Impossible events

D3.2.2 Probability of an event

D4 Judgement of equality in fairness

D4.1 Almost the same numbers on the scorers means that the game works/it is fair.

D5 Proportional thinking

D5.1 Equality of balls vs counting

D5.1.1 Many times children didn't count the balls to see if they are equal, but they created strategies to have the same amount of balls.

D5.1.2 Equality of an event: 1:1 = 2:2 =….=n:n ?

D5.1.3 Do they add or do they destroy balls to make both colours having equal numbers?

D5.2 Double vs proportional thinking

D5.2.1 They could control the equal and sometimes the concept of double but they confused on any other ratios. They deal with big numbers by using the words 'more' and 'less'. To make decisions on ratios the global event helps more than the local.

D6 Infinity

D6.1 They make changes in the game and the mechanism to get bigger numbers.

D6.2 Method of trial and error

D7 Possibility

D7.1 Everything is possible to happen. Extreme variability is possibility.

The codes of expressions of random mixture were used for the main analysis of the data and were looking at children's expressions of randomness. The codes were based on how children expressed the movement of the white ball in the game, what changes they did in the mechanism for expressing randomness, how they changed the variations of sample space, how they judged equality in fairness and how expressed proportionality, infinity and possibility. An example of a coded transcript is presented in Appendix 4. A snapshot of this appendix is as follows:

Line			Codes
119	J:	You see, now it means that they have the same size. Our space kid is	A2
		near the yellow line.	D3.1
121	R:	How long do we need to leave the game on?	
122	J:	Four seconds...may be...I don't know. We need to see. But now it	D6
123		moves quickly. The white ball goes many times on the circles (the	
124		big balls), because they are big. Now the blue has more points, now	D6
125		the red....the blue...the red.	
126			
127	R:	For how long do we need to leave it going?	
128	J:	We need more seconds...The red now is 91 & 80...	D6

For example, line 119 of Appendix 4 has been coded by the A2 (The use of global evidence in judgements of equality), D3.1 (Construction of Fairness) codes. Here, the child referred to the same size of the balls in the lottery machine to express fairness in her game, while she was also using the global event of the game (the space kid) to judge her construction. Moreover, lines 122-128 have been coded by D6 (Infinity) code, as she expressed her ideas of leaving the game to run longer, in order to be able to judge her construction and achieve fairness in her game.

The analysis of the data of the learning investigation phase is presented in chapters 6, 7, and 8. The following chapter, the Game Evolution will give a description of each iteration that took place up to the structure of the final game.

CHAPTER FIVE

The Evolution of the Game

5.1 Overview

This chapter outlines the evolution of the 'lottery game' through the three iterations of the research. The chapter refers to the Phase 1 (Iterative Design Phase) of the study (see section 3.2). It aims to show how the design of the game changed in response to the children's actions, and how children themselves responded to the game. The evolution of the lottery game took place over three iterations. I will describe the game as it evolved through each iteration, its problematic situations, the reactions of the children and how the problematic situations probed the children's probabilistic ideas.

5.2 Iteration 1 (Phase 1): The storm game

5.2.1 Description of the storm game

The storm game was written in 'Rule-Maker' software (see section 3.3.2). In this game the story was presented to the children as follows: 'A clown is cycling through a park. The day seemed to be great, but suddenly a cloud appears in the sky. The clown tries to avoid the storm, so it moves forwards and backwards in order not to be under the cloud. What will happen? Let's see... '(see Figure 5.1). The lottery machine consisted of red and green balls, placed in a linear position. The children could add and move balls of any colour from the lottery machine and shuffle them by clicking the mouse.

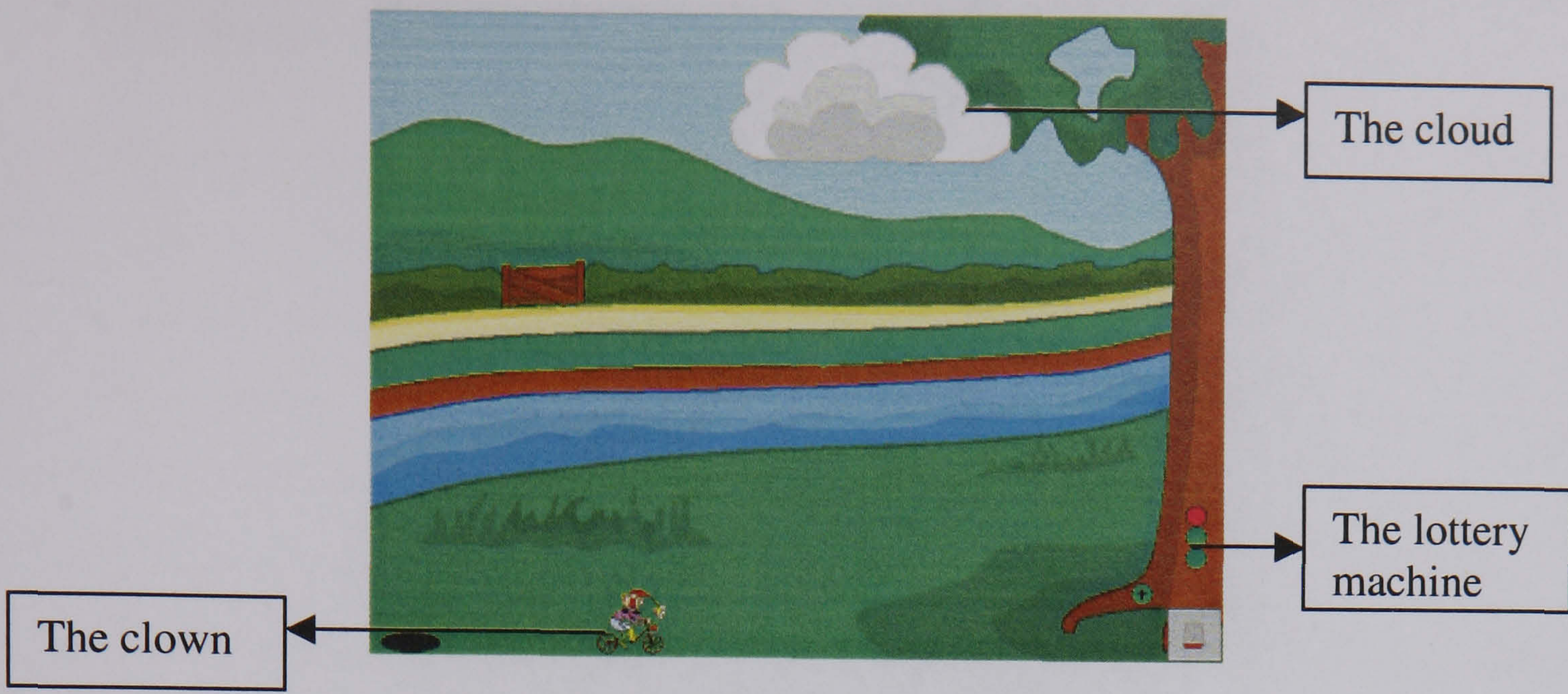

Figure 5.1: The storm game

5.2.2 Problematic situations of the storm game

There were four versions of the storm game: balls game, cloud game 1, dice game and cloud game 2. The problematic situations of each version were as follows:

Game 1: Balls Game

The task of this version was children to try to get the red ball to the top of the lottery machine. Secondly, they needed to predict what would happen in the next shuffle. Thus, the problematic situation here was to make manipulations in order to succeed to get a red ball at the top of the lottery machine. The children could add and take away balls from the lottery machine (see bottom-right corner on Figure 5.1). They shuffled the balls, by clicking on the lottery machine, and added and took away balls from the lottery machine with the aim of getting a red ball on the top. This activity gave the children the opportunity to express their thoughts about randomness through the role of the red balls in the game.

Game 2: Cloud Game 1

This was a game in which the behaviour of a cloud was governed by the rule 'if red is on the top then I make lightning…' Here, the children expressed their thoughts about the probability of an event, then they made probability comparisons and how these affected the scores in the game. The rules of the cloud are described in Figure 5.2. The cloud had the rule 'always move forward and if red is on the top then make lightning'.

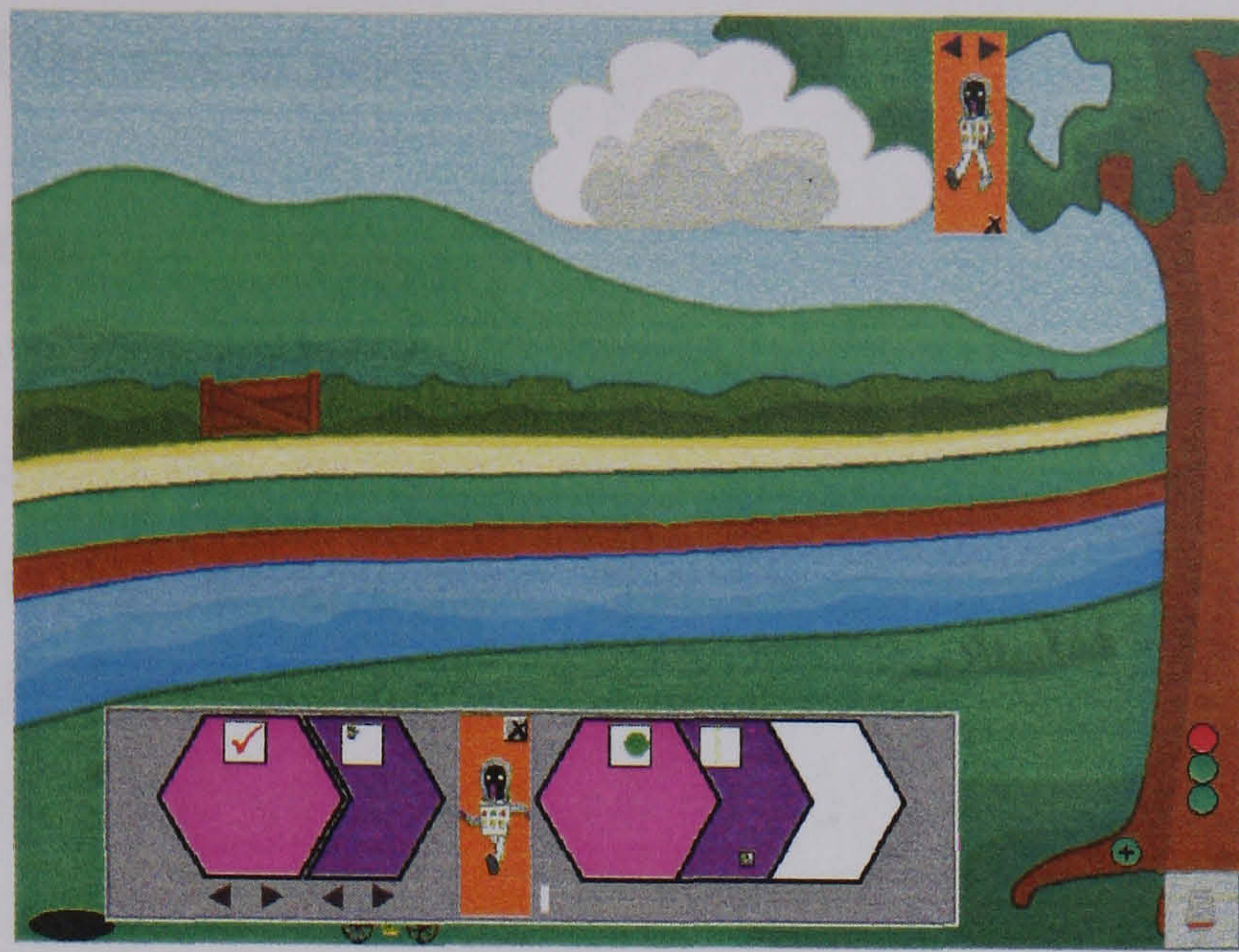

Figure 5.2: The actions and conditions of the cloud

In this game the children had to find out the rules of each object in the game in order to understand the connections between them. Furthermore, they were asked to make changes in the 'lottery machine' in order to construct a fair game, and to make more changes if their first ideas did not work. The children worked in pairs. One controlled a storm cloud, which was connected to a 'lottery machine'. The other child controlled a biker-clown, moving right and left, by using a joystick, trying to avoid being under the cloud. The child who controlled the cloud could make changes in the lottery machine in order to win. In the next round, the child who controlled the biker-clown could make changes to the lottery machine to avoid having too many strikes in the game. In this way, the children had to decide what changes they could make in order to make it easier for them to win.

Game 3: Dice Game

In this game the behaviour of the cloud was governed by the rule 'if thrown six…' This required the children to express the probability of an event with another 'machine'.

Game 4: Cloud Game 2

In this game the children were asked to change the rules of the cloud in order to be similar to those of the dice game. Hence, they had to compare the lottery machine with the dice machine.

5.2.3 Findings from the analysis of the iteration 1 data

Four children participated in this iteration. The number of children was intentionally small as the aim was to inform iteration 2 of the game design process, with the focus firmly on the design issues. The rationale for the analysis of the data focused on the design of the game rather than on the thinking of the children (for more details about the data collection in this iteration, see Chapter Four).

5.2.3.1 The design of the storm game and the expressed probabilistic ideas

The first design change that emerged from the 'game use' was the presentation of the lottery machine. The presentation of the lottery machine confused all four children. For example, Ellis[21] described the 'balls game' as 'the red ball goes in columns and it goes forward. It changes colour and position'. Both pairs, after playing for some time with shuffling the balls, were looking for patterns between the two colours of the balls:

Kate: We have a pattern. Red- green- red- red- green. Two patterns.

Jerry, in response to my question about whether at the next shuffle the red ball would go on the top, answered:

J: No! We saw that it goes down, up, down, middle, down, middle, down and up, down, up, down, middle, down, middle, down.

Researcher: Ok. Do you think that it always follows the same procedure?

J: Yeah. Because we saw it.

R: Do you think this will happen again?

Nichole: I think it will. Let's see....

J: Hey! What's wrong? Now it follows one pattern and another.

Jerry still believed that the balls follow a pattern that changes over time. After shuffling the lottery machine seven times she decided:

Jerry: No there is not a pattern.

Researcher: So, how do they work?

J: Like crazy.

But what confused me is that after this statement; she was still looking for the 'right pattern'.

Jerry: No, no...I got it.

And she tried to explain to Nichole what happened and how the pattern worked.

It was clear that there was a particular difficulty concerning the *linear representation* of the balls and that the desired result for a red ball to be on the top. Although the four children tried many times to add and take away balls, clicking on them to make them change places, the idea that the colour that they wanted should be the one on the top confused them, although it was explained by the researcher.

J: So, what are the green balls for?

N: Nothing. But, I have red on the top.

[21] The names are pseudonyms.

J: So, take a green and put it on the top.

...

J: The balls move behind each other quickly. (*She tries it out.*) No, they don't change places they just change the colour. The red becomes green and some greens red.

Jerry tried to find out how the balls were being shuffled. Because of the linear representation of the lottery machine, she seemed to be confused that the balls had just changed their colour.

All the four children understood easily that the actions of the cloud in the Cloud Game were connected with the lottery machine balls. It appears that this was based on knowing how Rule-Maker worked and the effect of 'if ... then ...' rules. Jerry's comment after exploring the cloud's robot was 'So, if we don't have any red here (on the tree), we do not have any strikes'. This rule gave the children the opportunity to connect the storm game with the balls game. Also the decision to use balls instead of dice made it easier for them to change objects (colours) and to count the number of events.

It can be concluded from this phase that the four children could easily manipulate the lottery machine and could mentally connect the objects of the game. In the game the children made predictions, constructed representations inside the lottery machine and tried to make things happen -all of these are key activities for learning according to Harel (1991). However, it seems that the linear representation of the lottery machine encouraged all the four children to look for patterns in the outcomes.

The children expressed *randomness* in the game in many different ways. They called the game 'moving balls', 'giggling balls', or even 'crazy balls'. The children found it 'hard' to be sure of their predictions. The 'icons' of the coloured balls made it obvious for them that equality of colours in the sample space generates *fairness*. This was something that the children did not appreciate when they played the dice game. A reason for this might be that with balls each outcome was represented by a different object (ball), but in the dice game it was difficult for the children to compare the different figures of the numbers.

It seems that the design choice to use coloured balls gave the children an opportunity to express ideas about impossible and certain events. Kate expressed the concept of *impossible event.* Kate needed green balls in order to have a possibility to win and she understood that the absence of red balls would prevent her playmate to play. Also, when

Jerry and Nichol made the rules of their game and Nichol put only red balls in the probabilistic machine, Jerry complained 'So, I will lose. It is not fair. The cloud will strike all the time'. That led Jerry to express the concept of *certain event.* There was a certainty of winning for her playmate and impossibility for her.

The use of coloured balls made it possible for the children to have as many events as they like without being obliged to think about numbers in order to express ideas about the *probability of an event* to occur. The children had to decide whether the red ball would be on the top, and Jerry realised that the probability of an event increased when that event appeared more often in the sample space. On the other hand, Nichol did not have a clear idea about this, although she seemed to realise that she had to change the quantity of the balls in the lottery machine in order to make it easier for her to win. At one point Jerry decided to put in her lottery machine only one red for Nichol. As she said 'There is a bit of fairness, but I have more (balls). I'm making it to win.' Later, she put as many greens as she could and she concluded that was very easy for her to win.

Although there was a direct connection between the objects of the game, it seemed that the design of the game made it less easy for the children to realise about the global events of the game, and this might be the reason for some difficulties in finding the probability of an event. There was an absence of 'continuity' in that they had to click on the mouse in order to find the next outcome of the game and this seems to have interrupted their connection between local and global events.

5.2.3.2 Summary of the design changes for the storm game

In iteration 1 the four children expressed some probabilistic ideas, but the game did not afford them many opportunities to express probabilistic ideas in the sense of Aim 1 in Chapter Three (iteratively to design a game to afford young children opportunities and novel ways to express and develop probabilistic ideas). Table 5.1 presents the key issues addressed for design changes in iteration 1and their rationale.

Iteration 1	
Key Issues Addressed for Design Changes	**Reason**
- Problem due to linear representation of the lottery machine. A different representation required	- This representation encouraged the children to look for patterns
- The children to be able to construct their own representation of a lottery machine	- The children would have the opportunity to express better their probabilistic ideas
- Having a continuous movement in the lottery machine (create a 'bouncing ball' using Pathways software)	- To develop a better understanding of the aggregate view of the game – i.e. the global outcomes of the children's lottery machine

Table 5.1: Iteration 1 design changes

The representation of the lottery machine needed to change for the next iteration, since the linear representation encouraged all the children look for patterns. This finding agrees with Ayres and Way's (2000) research in which students were also required to predict the colour of a next ball drawn and also inappropriately tried to utilise colour patterns as a strategy. What was needed was a way for the children not only to add and remove balls in the lottery machine, but also to be able to see an 'open structure' in the lottery machine. This should make the game even more 'dynamic' (see diSessa, 1995) and might avoid children looking for patterns.

Another key change for iteration 2 was to improve the 'continuity' of the lottery machine. The children had to shuffle the lottery machine by clicking the mouse and this action seemed to make it more difficult for them to understand the global outcomes of their lottery machine.

5.3 Iteration 2 (Phase 1): The space kid game (version 1)

Several attempts were made to satisfy the changes that iteration 1 dictated. This is one of the reasons that Iteration 2 could not take place until about a year later. Another reason was that in order to make the changes to the game there had to be changes in the software as well. A completely new version of the software was created, called 'Pathways', as a replacement for Rule Maker, and the iteration 2 experiments took place in April 2001. The children participating in this iteration were different from those who participated in iteration 1, but were also part of the Playground Project. The description of the children involved in iterations 1 and 2 has been already given in Section 4.2.1.

5.3.1 Description of the space kid game

The 'Space Kid' game, which was the focus of iteration 2, can be seen as a form of 'random walk'[22]. The random walk in the game involved a 'space kid', which moves upwards and downwards on a yellow line. These movements occurred one step at a time, in response to an outcome generated by the lottery machine. In it a small red ball bounced and collided continually with a set of static red and green balls, controlling the movement of the space kid. As programmed initially, a collision with a green ball added one point to the green score and moved the space kid one step down the screen. Similarly, a collision with a white ball added one point to the white score and moved the space kid one step up the screen.

The rules of the game are shown in Figure 5.3. A red ball moved inside the lottery machine at a certain speed and collided with green and white balls. When it touched a ball, this ball sent a message to the scorer of its colour and a message to the space kid. The space kid moved towards a 'planet' (or exploding mine). The rules[23] of the planets/mines were 'when I touch the space kid, I blow up and switch off the game'.

[22] Clapham (1990) describes:

> 'Consider a *Markov chain* X1, X2, X3,...in which the state space is {..., -2, -1, 0, 1, 2, 3, ...}. With the integers positioned as they occur on the real line, imagine an object moving from integer to integer a step at a time. That is to say, from position i the object either moves to i-1 or i+1, or possible stays where it is. Such a Markov chain is called one-dimensional **random walk**: if $X_n = i$, then $X_{n+1} = i-1$, i or $i+1$. The state i is an **absorbing state** if, whenever $X_n = i$, then $X_{n+1} = i$; in other words, when the object reaches this position it stays there. When a gambler, playing a sequence of games, either wins or loses a fixed amount in each game, his winnings give an example of a random walk.' (p. 235).

[23] The idea of rules in Pathways was described in section 3.3.3.

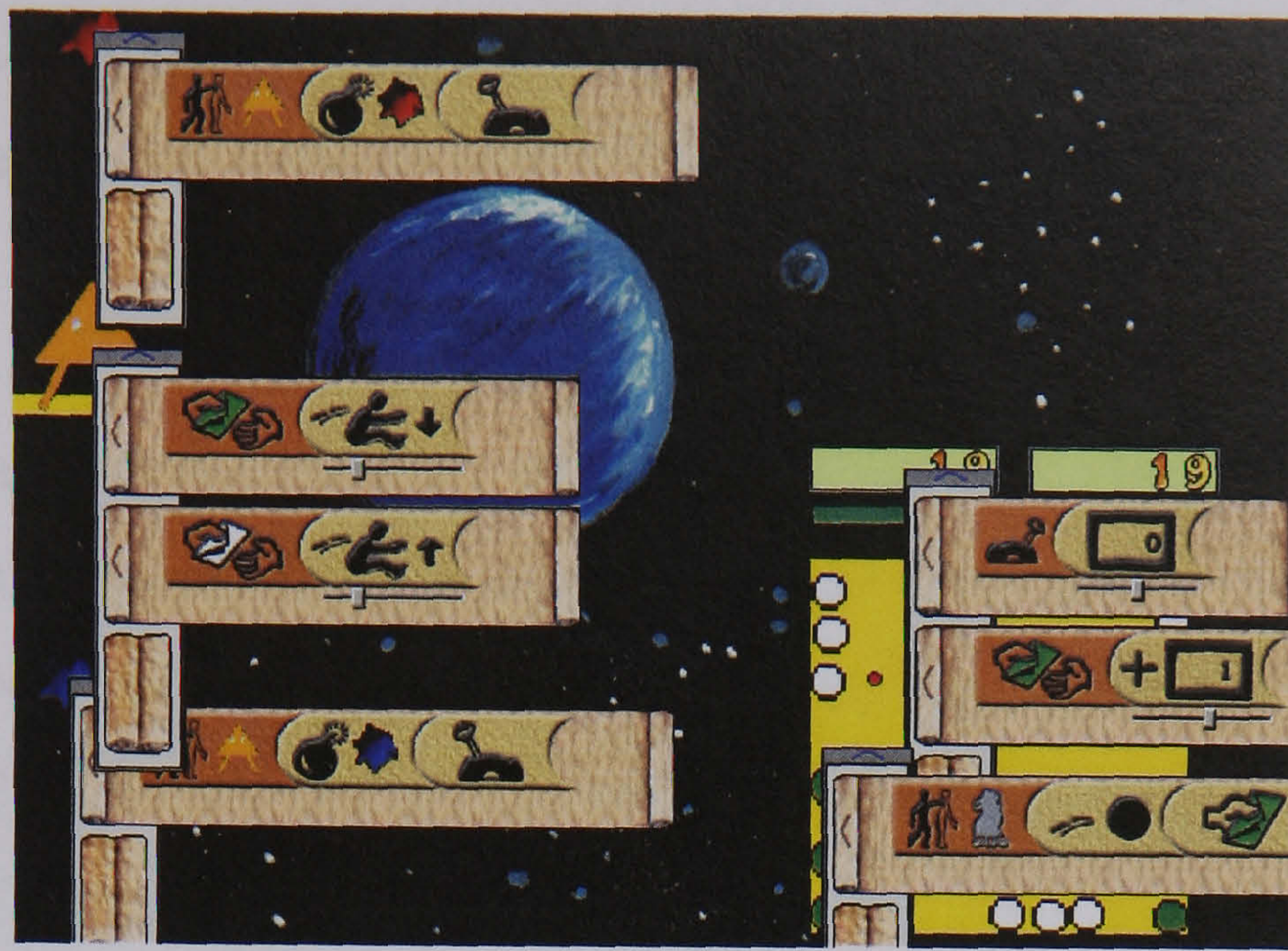

Figure 5.3: The rules of the space kid game, iteration 2

5.3.2 Problematic situations of the space kid game

The problematic situations that the children faced in this lottery game were presented as questions. The first question relates to fairness:

'What might you do if you want the space kid to stay near the yellow line while the game runs?'

The others refer to the probability of an event:

'What might you do in order to make the space kid reach the blue planet/mine?'

''What might you do in order to make the space kid reach the red planet/mine?'

The children were asked to try out their decisions, describe what happened, why they thought it had happened and if it did not work as they wanted, what else they could change.

5.3.3 Findings from the analysis of the data

In this iteration it was planned to have eight children, but after the first two task-based interviews, some immediate changes were indicated. So, in the remainder of this iteration only two children participated (for more details see section 4.2.1). As in iteration 1, the focus was on the design of the game and not on the thinking of the children, so that in-depth study of a small number of children was appropriate for the research.

5.3.3.1 The design of the space kid game and the expressed probabilistic ideas

Both children, Lisa and Sam, found the context of the game interesting and they found it easy to read the rules. This seemed to be a result of their previous experience with the

Pathways software. As had been hoped, the red ball bouncing inside the lottery machine (the major change from the previous iteration), offered continuity to the outcomes, and also helped the children not to focus on patterns in the outcomes as happened in the previous iteration.

Another point that emerged from this iteration was that one child (Lisa) wanted to have two different sample spaces at the same time, one for one colour of balls and one for the other colour. As will be seen later, this suggested designing 'bricks' to sub-divide the space of the bouncing ball in the lottery machine. What Lisa tried to do was to copy another bouncing ball, but because of the fact that the red balls did not have any rules, except to move with a certain speed, her idea did not work.

L:	That (one bouncing ball) is to be for the white ones and that (another bouncing ball) for the green ones...eh...
S:	It's not going to work. There is not a particular space for each one to move. This will touch the white balls as well and this one green ones, as well.
L:	No, no! But. If we just make another rule...oh...if I spread these around.
Researcher	What are you doing now?
L:	The green balls will go up until here. The others to be on the opposite.
R:	Oh! That's a good idea! Why is that?
S:	The white is going to have more.
...	
R:	So, did the space kid go back to the yellow line?
L:	Yeah!
R:	Oh...that then works ok!
L:	Yeah, but this one (the one red ball) went everywhere and that one (the other red ball) went everywhere.

Lisa wanted to create a rule where one bouncing ball would send a message only if it touched a particular colour. Unfortunately this could not work because of the absence of an appropriate 'stone' in the Pathways software. It needed a rule to say 'If I touch a particular colour, I bounce off and ....'. The issue of separate sample spaces was solved in the last iteration by adding bricks to the game design (see section 5.4).

Both children were asked what they would do if they wanted the space kid to stay near the yellow line. The idea of this question was for the children to construct a random sample space where white and green balls would be selected with the same frequency. This was achieved either by constructing a patterned spatial representation or by making a circle with different colours so as to hold the red bouncing ball inside it. Lisa also considered the possibility of the red ball escaping the circle, so she destroyed every other ball that was in the lottery machine and not in the circle.

The movements of the space kid, up and down, influenced Sam's construction for the sample space. Sam separated the two colours depending on the message that they gave. The white balls, which gave the message to the space kid to go up, were put on the upper level of the lottery machine and the green balls on the lower level. Sam however noticed that his lottery machine did not work, as he wanted, because of gaps between the balls. That was because the red bouncing ball stayed longer on the white balls, which made the space kid on average to move upwards. As Sam noticed, by having no gaps between the balls, the red ball would not stay for a long time in one group of colours. Thus, he achieved the wanted result.

The idea of certain and impossible events was easily expressed when the children were asked to make changes in the sample space in order for the space kid to reach the blue or the red planet. It seemed that the spatial representation of the lottery machine gave Sam the opportunity to construct a fair sample space and to express ideas of fairness, impossible and certain events. The two children related the idea of randomness and the probability of an event with *'a feeling'*. Their intuition about whether the space kid will go up or down was (see section 5.3.1) a different way to express the probability of an event. It seems that the 'feeling' became more certain when their construction worked for a long time. Lisa claimed that their game was *still* working, so it was a 'good construction'. The expression of connecting the effectiveness of a construction with the time for which the game ran occurred a number of times later.

5.3.3.2 Summary of the design changes for the space kid game

The game seemed to be interesting for the two children and it probed many probabilistic ideas. However, there were some significant changes in the software and in the game that needed to be implemented before the next iteration (see Table 5.2). A first immediate design change required was to work around bugs the software for using sounds. Another bug that had to be solved immediately was in the sending and receiving of messages. After a few minutes of showing and reacting to colours/messages synchronisation was lost between the receivers of messages, the score counters and the space kid (although the scorer added a point when they received a message the space kid did not move immediately).

Iteration 2	
Key Issues Addressed for Design Changes	**Reason**
- Sending and receiving bugs to be fixed	-To improve the link between the appearance of local and global events
- Change colours of the balls to indicate the movement of the space kid	-To have consistency between the colours of the balls, the colour of the messages, the mines/planets and the movement of the space kid
- Design a 'brick' object	- To be able to divide the lottery machine 'sample space' if required

Table 5.2: Iteration 2 design changes and their rationale

Table 5.2 indicates that a first key issue addressed for design changes was to solve the bugs in showing and reacting to messages. Also some 'bricks' were needed for the children to divide the lottery machine into different parts, so as to manipulate the sample space and distribution in more diverse ways. Furthermore, the colours of the balls needed to be changed. The planet/mine and the balls that control space kid's movement to this planet/mine needed to be the same colour, in order for the children more easily to connect the movement of the space kid with the collisions in the lottery machine. By the end of this iteration I also decided that in the final iteration, children should work individually with the task, since many times, in iteration 1 and iteration 2, some children were shy and hid their thoughts and decisions behind the other child's.

5.4 Iteration 3 (Phase 2)–Final Game: The space kid game (version 2)

Iteration 3 took place in October and November of 2001. This was the final iteration, used to collect the final data of the study.

5.4.1 Description of the space kid game (version 2)

The game for this iteration was similar as that in iteration 2, with the changes and corrections to the Pathways software carried out. The game, as in iteration 2, involves a 'space kid' moving upwards and downwards on a yellow line with planets/mines at the ends. The colours of the balls in the game had been changed and the look of the lottery machine for this iteration is illustrated in Figure 3.9 (section 3.4.3).

5.4.2 Problematic situations of the game

The final game consisted of four steps. The problematic situations were used as a base line for the researcher to probe the interaction between the child and the task, with the researcher shaping the interview based on the actions and responses of each child.

Step 1:

- On the screen: The space kid, lottery machine, the bouncing ball, one blue and one red ball, the scorers of the balls (see Figure 5.4).

Figure 5.4: The step 1 starting point of the space kid game in the final iteration

- Goal: Make the space kid move up and down.

Step 2:

- On the screen: As in phase 1, plus two planets/mines, one blue and one red a short distance above and below the space kid. There are more balls in the lottery machine. (see Figure 5.5).

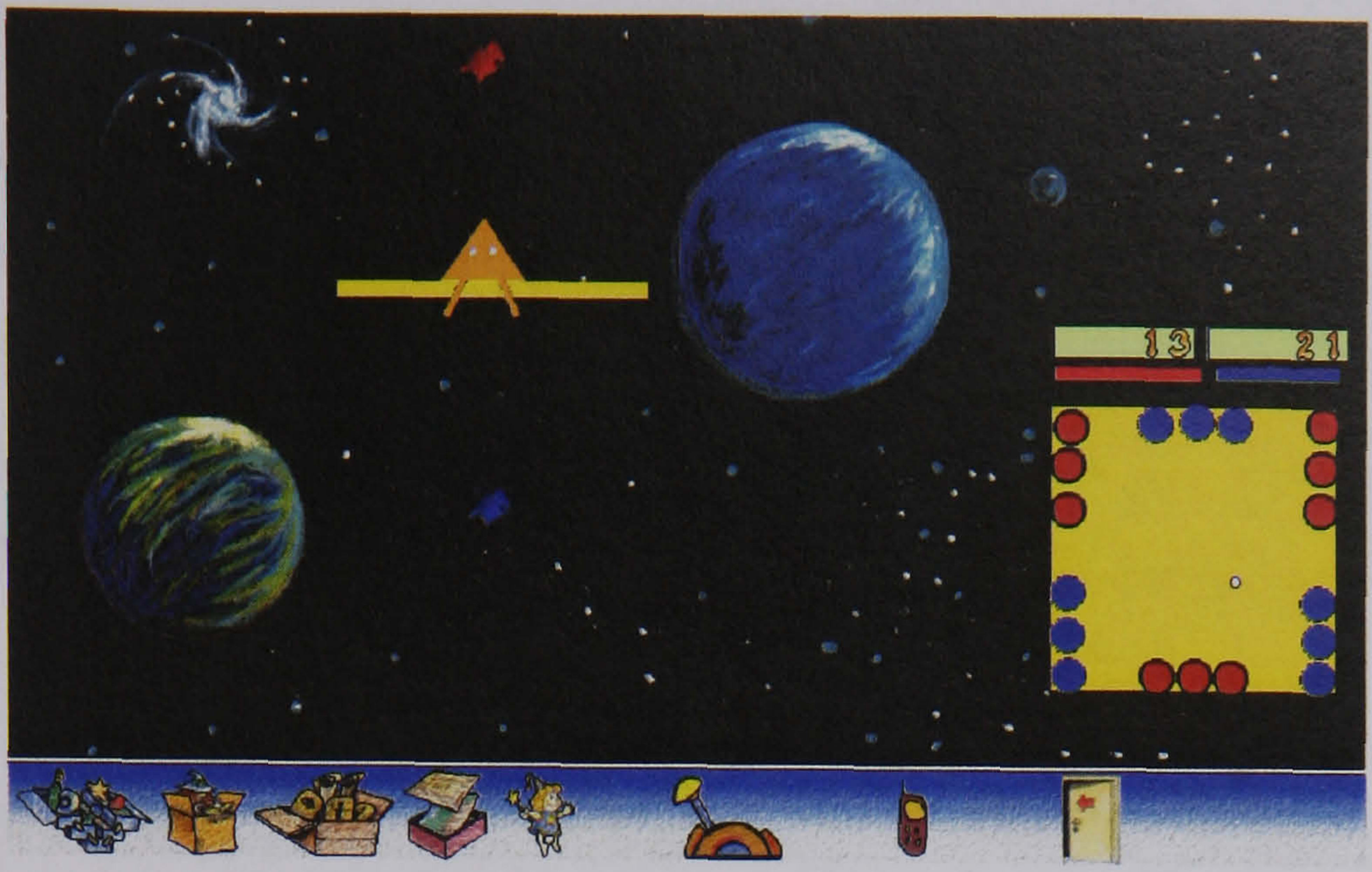

Figure 5.5: The step 2 starting point of the space kid game in the final iteration

- Goal: Make changes so that the space kid stays near the yellow line and does not touch the two planets/mines.

Step 3:

- On the screen: As in step 2 plus two white balls bouncing around. A brick object (a light brown rectangle) is available that can be copied, moved and used to divide the lottery machine space (see Figure 5.6). The brick has a rule 'If I touch something, it bounces off me…'

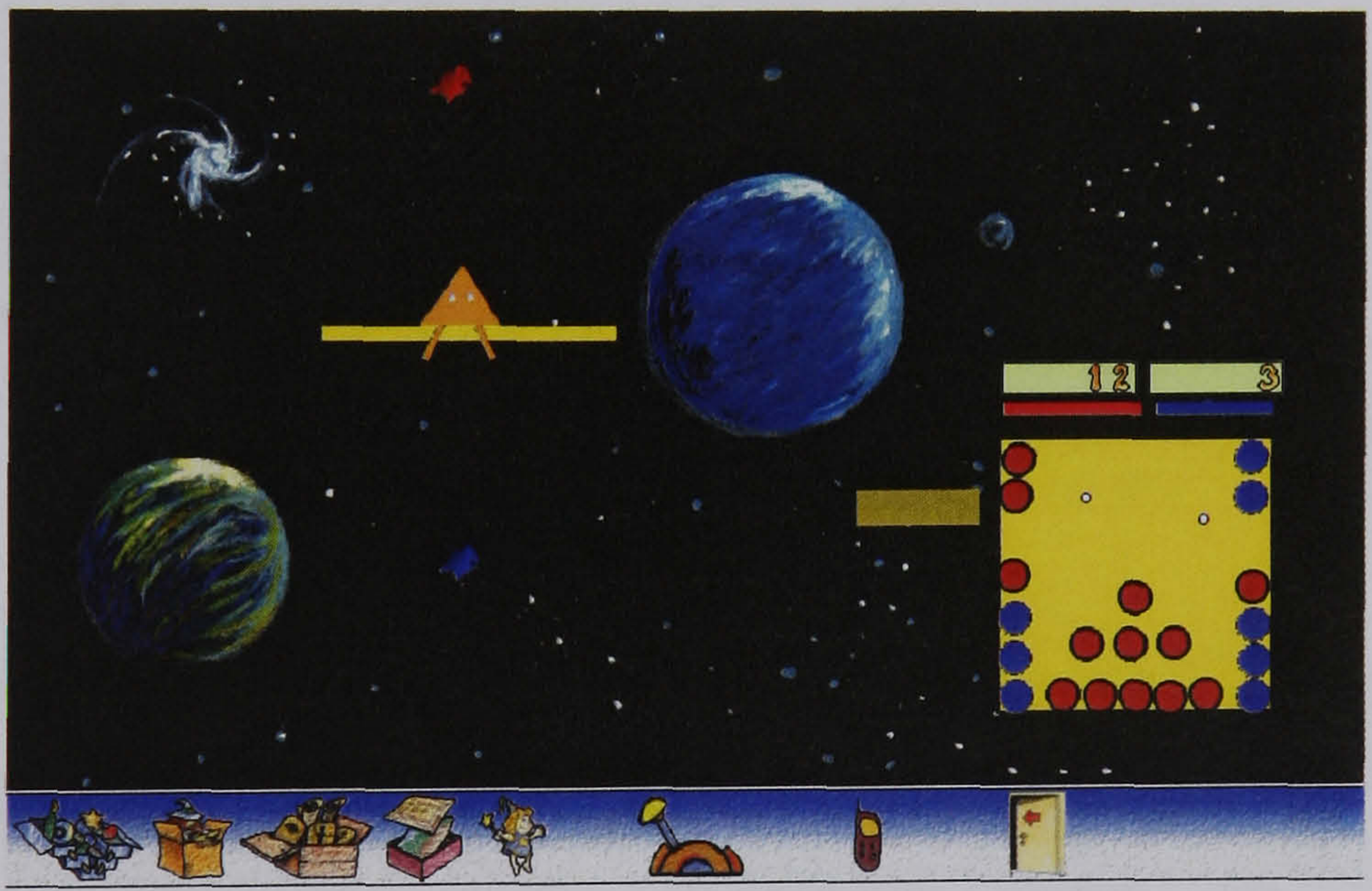

Figure 5.6: The step 3 starting point of the space kid game in the final iteration

- Goal: Make changes to the lottery machine so that: a. the space kid stays near the yellow line b. the space kid touches one of the two planets/mines.
- Probes by researcher: How do children express their ideas about randomness, fairness, certain and impossible events, probability of an event, proportional thinking.

Step 4:

- On the screen: As in step 3.
- Goal: To build a new probabilistic game.
- Probe: How do children express randomness?

The tasks of each step are described in the protocol of the task-based interview — see Appendix A3. The data collection of this iteration has been described in section 4.3.3.

5.5 Summary of the Game Evolution

The evolution of the game took place in three iterations. Table 5.3 shows the suggested changes of each iteration.

Iteration No	Iteration 1	Iteration 2	Iteration 3
Changes needed for the next iteration	- Avoid the use of linear representation of the lottery machine - Children to be able to construct the spatial arrangement of the lottery machine - Have a continuous movement in 2-dimensions in the lottery machine (create a bouncing stone using Pathways software)	- Fix the sending and receiving bugs - Change the colours of the balls to indicate the movement of the space kid - Design a 'brick' object for separating the lottery machine space if required	Final iteration: Satisfies the aims of the study

Table 5.3: The suggested changes for each iteration

Iteration 1 suggested changing the linear representation of the lottery machine, as this seemed to influence the children's predictions. It also suggested the need to have a continuous movement in the lottery machine, as the absence of this did not help children to link effectively the local and global events of the game. Iteration 1 also suggested the possibility for children to construct not only the number and colour of the lottery machine's balls, but also the whole structure of the machine. This was the reason of the constructing a 2-d spatial representation of lottery machine, which can be characterised as the key finding of Phase 1, iterative design phase. The children could make changes to the two-dimensional continuum space to make changes to the global events of the game. In this context, without this characteristic, the children were generally unable to express notions of random mixture.

Iteration 2 satisfied the iteration 1 changes, as far as it concerned the construction of a 2-dimensional continuous spatial representation of a lottery machine. In contrast of the lottery machine in iteration 1, children in iteration 2 seemed able to express the notion of random mixture in two-dimensional space continuum environment. The computer environment supported several ways in which the idea of mixture could be expressed. The children changed the position, size or number of the red and blue stable balls in the sample space, or they described in words the 'uncontrolled' continual movement of the white ball.

Iteration 2 also indicated some further changes to the design of the game. It indicated some bugs/problems in the Pathways software and also suggested making consistent the colours of the balls to be the same as the colour of the planets/mines. This helped children to link the movement of the space kid with the selection of the coloured ball in the lottery machine. Iteration 2 also suggested the possibility of having some tools (bricks) for children to manipulate and divide the space of their lottery machine. Iteration 3 implemented the iteration 2 changes and was judged to satisfy the aims of the study. Thus, it was regarded as the final iteration for the main data collection; and the data from this will be analysed in the next three chapters.

The analysis of the final data will be described in the following three chapters. Chapter Six will analyse how children's thinking moved from sample space to global outcomes, Chapter Seven will explore children's understandings of fairness and Chapter Eight will examine children's quantitative ideas of randomness.

CHAPTER SIX

Linking Local to Global Events

> 'The space kid will be in the middle, may be little up or little down, it depends on the white ball. You see! It (the white ball) moves by itself. It goes to different places and if it goes down and there is a red ball there, it touches it and our space kid moves up.'
>
> (Jane, 6 and 7 months year-old girl)

6.1 Overview

This chapter is part of the Phase 2, learning investigation phase, and it analyses data from iteration 3, the final iteration. This involved children working with the 'Space Kid' game as described in Chapter Five. Twenty-one of the twenty-three children initially focused on the movement of the white ball inside the yellow square. The first part of this chapter considers the continuous 2-dimensional movement in the lottery machine and describes children's ideas about the movement of the white ball. The second part considers how the position of the white ball influenced children's ideas, and the last part considers children's ideas about the connections between local and the global events in the game.

6.2 The continuous movement in the lottery machine

As Jane's description above shows, the children saw a connection between the space kid and how the white ball moves. The analysis of the data presented here was mainly based on Code A: Local and Global Thinking (see Chapter Four, section 4.3.5), which referred on the children's connections between the local events of the game and the global ones. Jane describes this arbitrary movement by saying that the ball moves by itself, going to different places. Jane seemed to be interested in watching the *arbitrary* movement of the white ball, which moves around into different places. She also seems to understand that this affects the space kid's movement, which in a fair game should stay in the middle, near the yellow line.

Rachel (7 3/12 year-old girl) is one of the twenty-one children who initially focused on the movement of the white ball rather than focusing on the movement of the space kid. She focused on the lottery machine, trying to impose on the white ball some kind of determined and predictable movement. The analysis here was based on code D1: Unsystematic Movement (see section 4.3.5), which refers to the children's descriptions of the movement of the white ball. However, the continuous movement of the white ball did not allow her to predict the movement of the ball. In the following snapshot Rachel's first reaction is described, after she made her first change in the sample space.

Rachel: The scores will change, now. The blue will get fewer points because the white ball hardly ever goes to the corners.

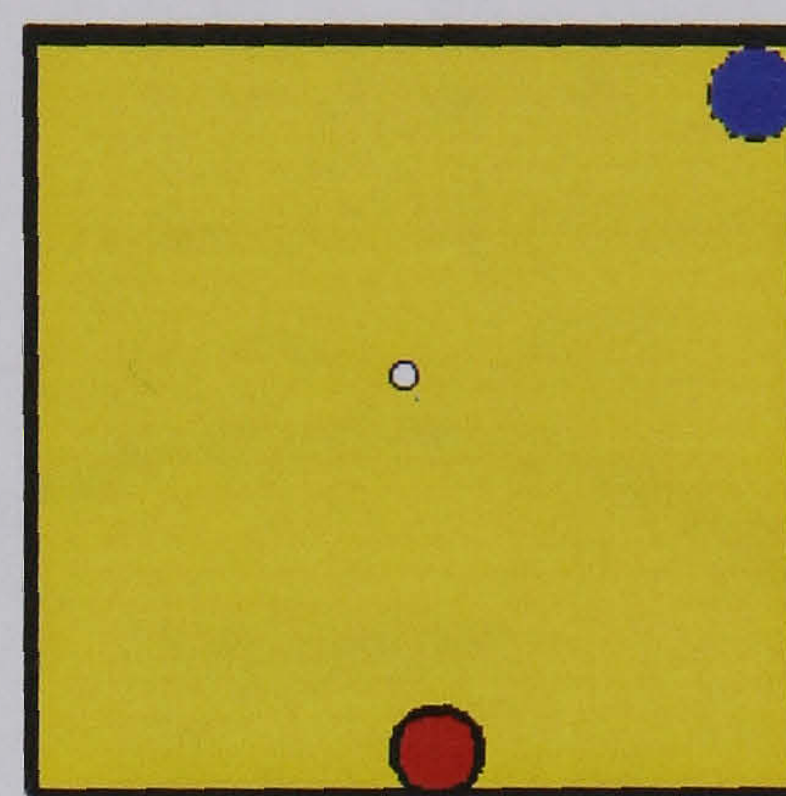

Figure 6.1: Rachel's first construction

Researcher: Why is that?

Rachel: The white ball moves like this both here and there. It moves up, down, right, and left. (*She is indicating the movement on the screen, by avoiding the corners).* I don't think that it goes to the corners… Let me try to see.

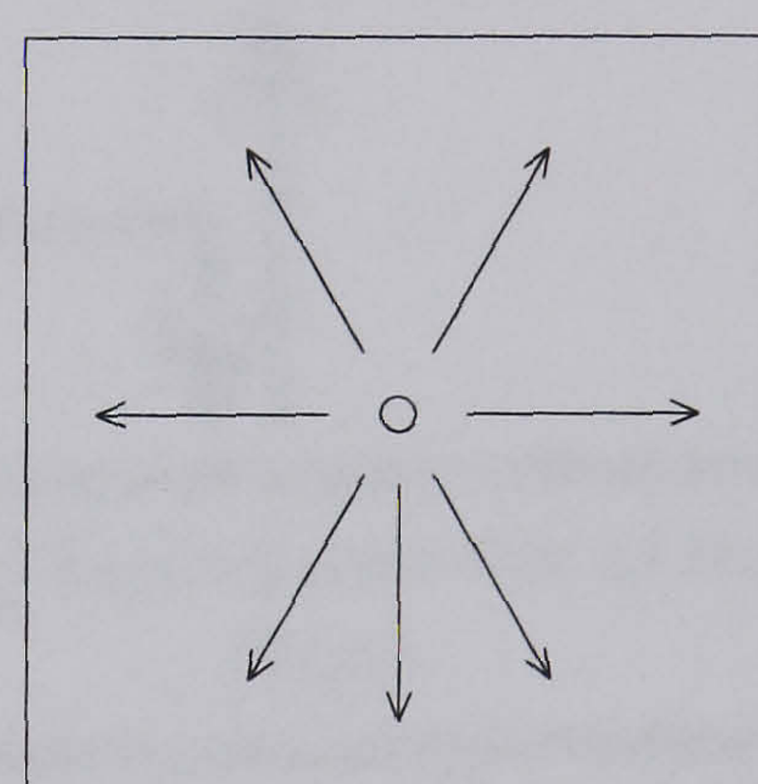

Figure 6.2: Interpretation of Rachel's description of the movement of the bouncing ball

She starts the game.

Rachel: You see, I was right! Oh! No! One point for it, another one, oops!

Researcher: How does the white ball move?

Rachel: It moves **where it wants to**…I mean if it is here (*she is pointing on the screen*), in the middle, it might go here and here, and here, **everywhere**. Look we have equal points now!

At first, Rachel explained the ball as 'deliberate', avoiding the corners that the white ball 'hardly ever moves to the corners of the square'. In fact, a more general observation is that, like twenty-one other children, Rachel's explanation of randomness focuses on the movement of the white ball in itself, rather than, for example, on when the coloured ball is touched and with what frequency. But, in the above episode Rachel realises that the ball 'moves where it wants to'. It seems that the aggregate view of the movement of the ball helped Rachel to link local events derived from randomness (the movement of the white ball) with a global understandings (the movement of the space kid). Also, the continuous movement of the ball helped her to link the short-term movements of the ball with its long-term movements (see Chapter Three, section 3.4.1). The following diagram shows how the medium helped Rachel to link the different levels of randomness (see Diagram 6.1).

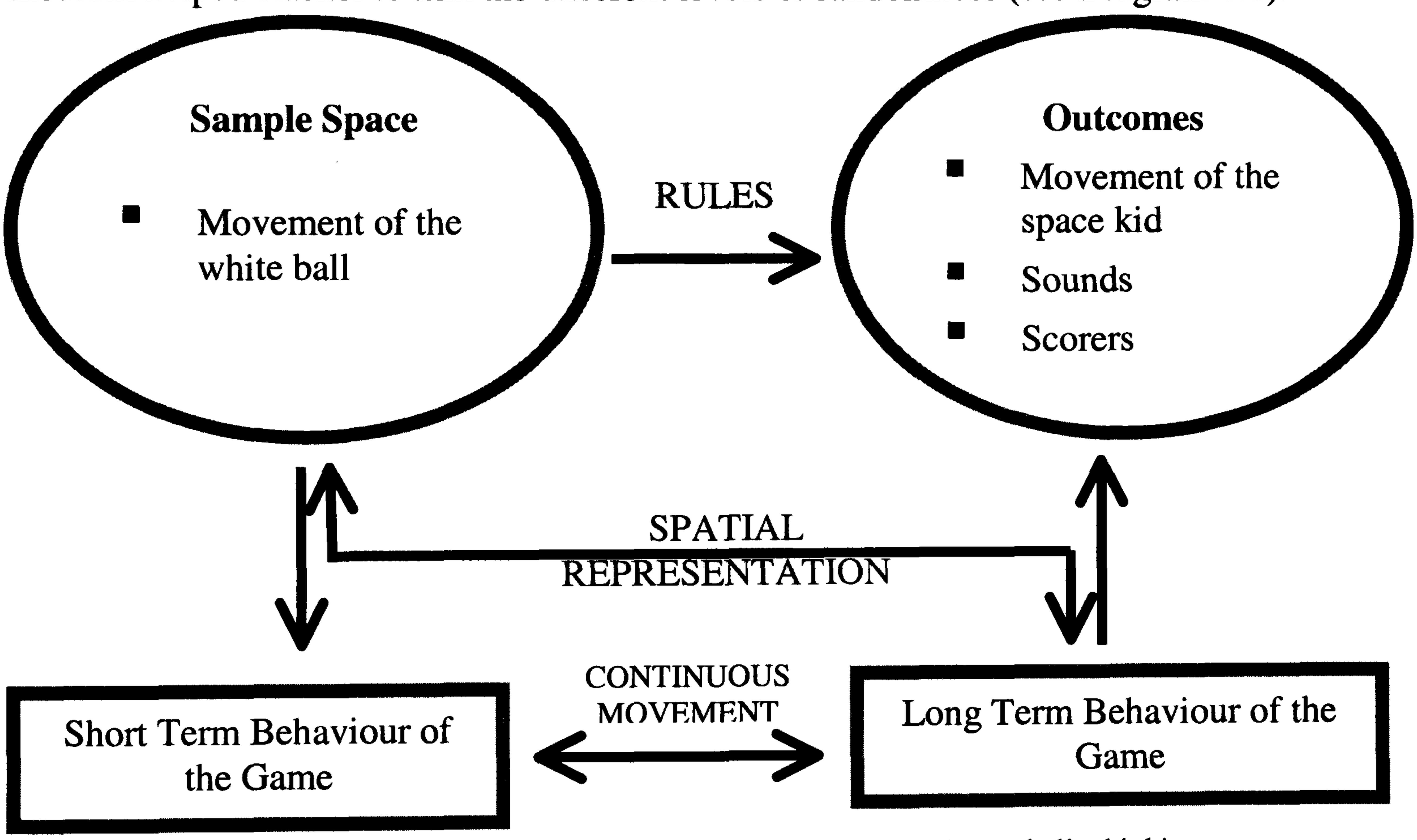

Diagram 6.1: How the task linked the parts of the game in Rachel's thinking

Diagram 6.1 illustrates how the rules of the game linked the sample space to the outcomes in Rachel's case. The rules linked the movement of the white ball with the movement of the space kid, the movement, sounds and the scorers, which represented the outcomes of the game. The representation of the two-dimensional continuous lottery machine made it possible for her to see the movement of the white ball not only as a short term behaviour of the system, but also as a long term behaviour. The data showed that the continuous movement of the white ball in the game helped Rachel to connect the short-term movement of the white ball to its long-term movement. This event played a major role in connecting local and global events and Rachel's description shifted many times from local to global events. A snapshot that shows this shift is the following:

Rachel: I will destroy some balls…

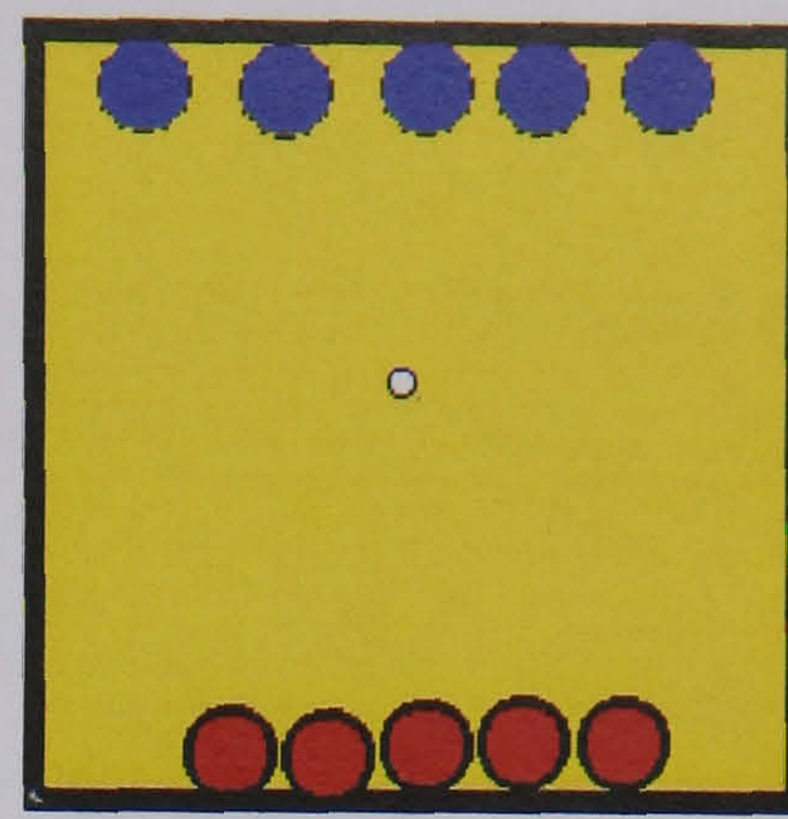

Figure 6.3: Rachel's construction of fairness

She starts the game.

Rachel: Did you tell me to be on the yellow line? (She laughs). Oh… it moved to the blue planet.

Researcher: Why was that?

Rachel: Because the ball moved like that and touched the blue balls… I don't remember very well how it was moving…When it touched a ball it changed direction …I don't know how it moved around.

At first, Rachel tried to guess the initial movement of the white ball and make a decision about how the ball would move and which balls would touch more frequently. Her first decision worked on the short-term movement of the ball, but not in the long term. However, the continuous movement of the white ball in her lottery machine made her 'not to remember', as she said, the particular way the ball was moving. This was the reason that she finally admitted that she did not know how the ball was moving around.

6.2.1 Short-term movement

The continuous movement of the white ball could be categorised in short-term and long-term movement. Twenty-one children out of twenty-three initially described a short-term movement of the white ball. Short-term movement refers to the exact previous movement of the white ball. Like Rachel, Zeta (6 4/12 year-old girl) also tries to find a particular way for it to move.

Researcher: How is this little ball moving?

Zeta: By hitting things.

R: Do we know where will it move?

Z: If it is here and moved there it will move there and touch this ball here and then here. (*She is pointing on the screen).*

Zeta was pointing to the short-term movement of the white ball, based on her previous experience of watching the ball moving around. This short-term movement of the white ball is also obvious in Simon's (7 10/12) construction.

Simon: First of all I should move this ball and get the star and I will do it like this!

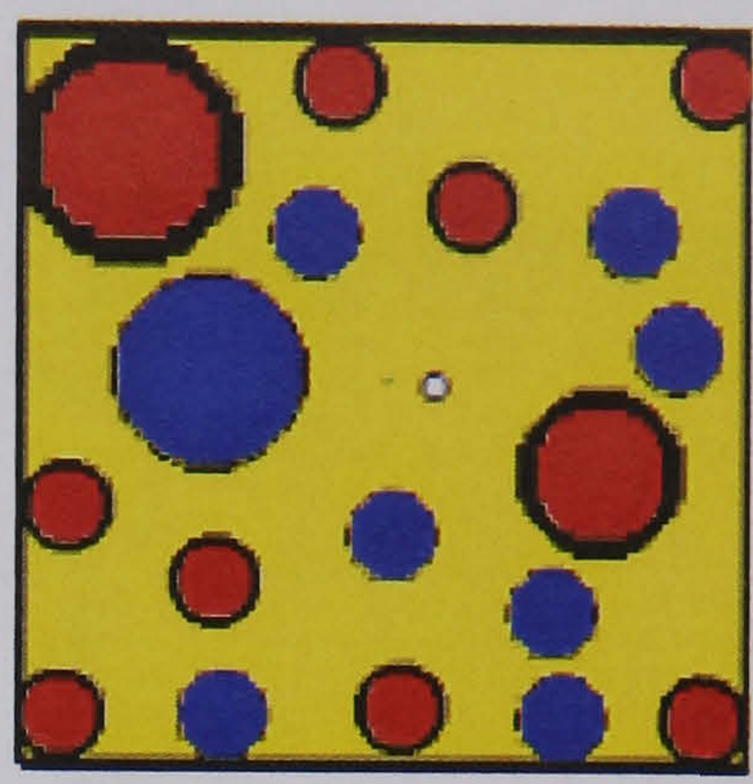

Figure 6.4: Simon's construction based on the short-term movement of the ball

Researcher: What will happen now?

S: First it (the white ball) will move to the big blue ball and then to the big red one and then to the two blue balls and then to the two red balls and then again to the blue and then again to the reds…

Simon placed balls in the sample space, having in mind a particular movement of the white ball. It seems that the predictable reflection of the white ball when it struck a coloured ball, made Zeta and Simon believe that the prediction of the white ball's movement was possible. This idea of reflection might be why Lucy (7 8/12 year-old girl) tried to make the white ball bigger in order to be more obvious for her where it would bounce and reflect.

Lucy: Ok. That's fine. I will make this bigger to make it move from one side to the other.

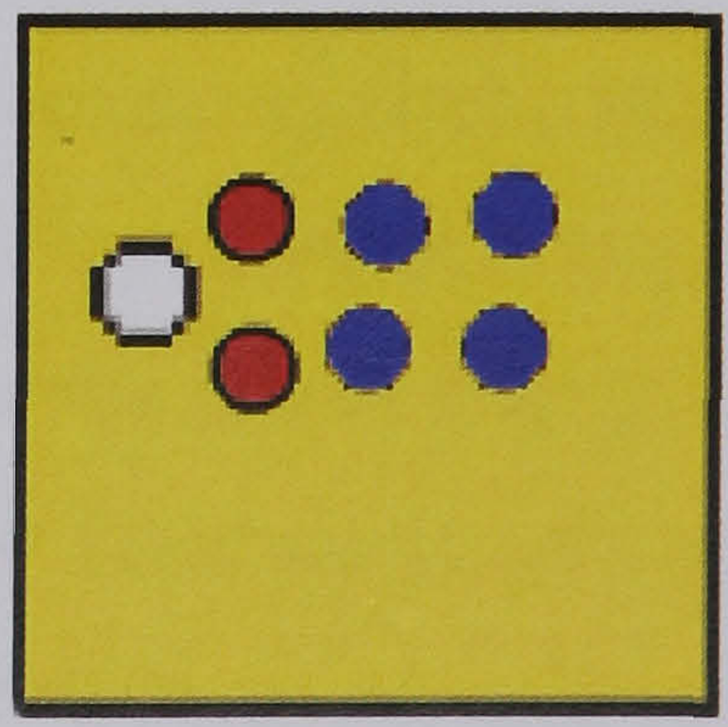

Figure 6.5: Lucy's construction based on reflection

Lucy here made the white ball bigger to be easier for her to understand where it would move and which ball would touch.

Twenty out of twenty-three children based their constructions on this idea of reflection in the short-term movement of the white ball. They constructed a 'fair' space with two separate 'teams' of balls one at the top and one at the bottom so that the white ball will bounce on one side and reflect to the other. For example, John's (6 10/12 year-old boy) construction shows that he thought of placing two teams, up and down, in the sample space.

J: It (the white ball) will move once up and once down…but…this (the space kid) will not move very far away from the yellow line.

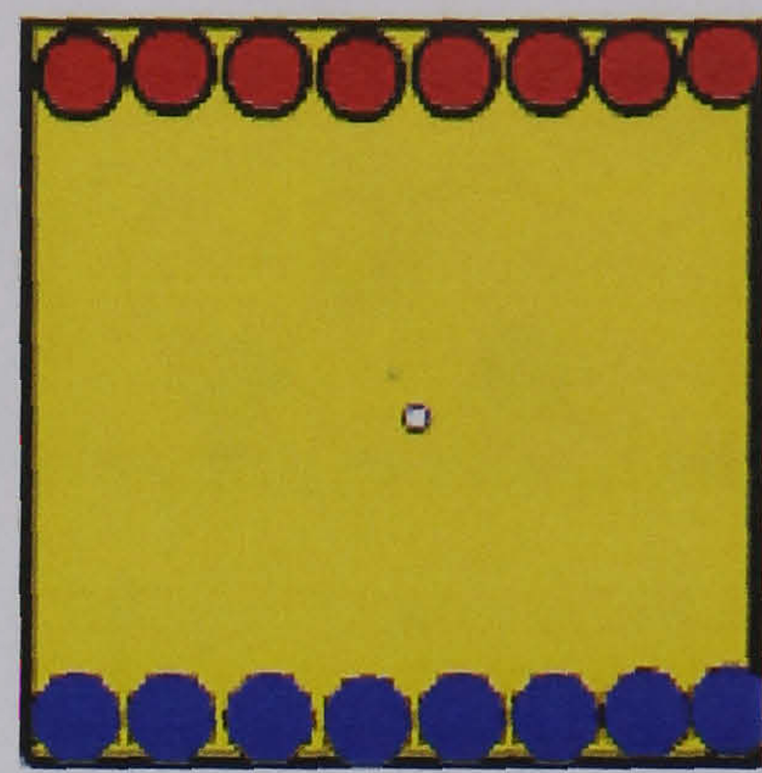

Figure 6.6: John's construction based on the movement of the white ball as bouncing ball

Although John's idea worked for his space kid to stay near the yellow line, most of the children's ideas based on the short-term movement of the ball did not work properly and did not bring desired result in the game -for example Simon's idea (see Figure 6.4). Thus, all the children had to change focus to the long-term movement of the ball, as described in the next section.

6.2.2 Long-term movement

The long-term movement of the white ball refers to the aggregated movement of the ball. The twenty-one children's gradual awareness of the lack of a pattern and lack of control over the movement of the ball encouraged them to change focus to the aggregate, long-term movement of their ball. Lucy characterised the white ball's movement and mixture as follows: 'it goes right-left, up, down and on the balls. We don't know where it goes. It moves in the yellow square, where it wants to go'. 'Mixture' for Lucy meant that the movement is where the ball 'wants to go', without any obvious pattern or pre-ordained paths.

An example that shows how continuous movement played a role for the twenty-one children to link the short-term movement of the white ball to the long-term movement is the case of Victoria (6 6/12 year-old girl):

Researcher: How did you arrange the balls?

Victoria: If it (the white ball) goes like this it might get the red (ball) and then the blue (ball) and then to move down like this or it may move like this and move up and get this one and then that one or to move in a different place and get this and this.

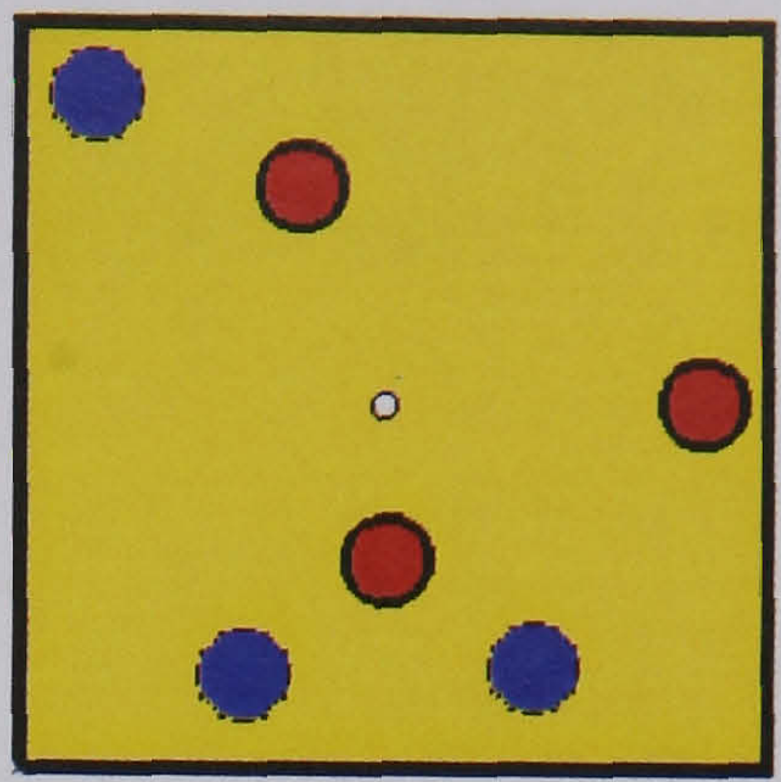

Figure 6.7: Victoria's first construction based on the movement of the bouncing ball

She starts the game.

V: Come on! Look! It (the white ball) gets red again! Now it (the space kid) moves down and then up and…Whatever I say it happens! Come on…It (the space kid) moves up and then down. Oh…no, we have more points for the red colour. I wanted to get one red and one blue. Go up, now move down…

R: Does it listen to you?

V: No! I will place somewhere else the white ball. Here! Let me start the game again.

V: Come on… (*she knocks on the table*), come on…Oh, not again. I can't control this white ball….

She stops the game.

Here, Victoria based her ideas on the previous movement of the white ball, thinking that it would follow the same path. As the game progressed, she began to realise that even if she was able to predict where the white ball ended up, she could not predict how it might get there (i.e. its path of movement), and she could not control or predict exactly its next move. Finally, Victoria focused on the movement of the space kid instead of the white ball.

Zeta argued that the white balls '… were touching balls and when they touched them they were moving. I did not manage to control them'. The 'unmanageable' movement of the white balls was one of the children's ideas about randomness. Helen (7 6/12 year-old girl) said 'The white ball moves randomly...but we want to touch the balls in order to get the same points for blue and red scorers'.

Therefore, the following section the ideas of this 'arbitrary' movement of the white ball will be described, as this arbitrary movement is linked to randomness in the game and it will help us to understand better the children's thinking about randomness in general.

6.2.2.1The arbitrary movement of the white ball

The arbitrary movement of the white ball presented a mystery for all the children. After Jane (6 7/12 year-old girl) played the game several times she attempted to describe the movement of the ball:

Researcher: How is the white ball moving around?

Jane: It moves up and down, right and left and when it touches one ball and gets a point it then goes everywhere in the yellow square.

R: Does it know where to go?

J: It knows.

R: How does it know?

J: It does know, we don't, but **it** knows.

R: How does this happen?

J: For us it moves randomly, but it **might know where to go**. We don't know where it goes because we didn't make it to go somewhere...

Jane seemed to understand that the movement of the white ball is arbitrary. She makes a distinction between the people who play the game, including her, and the computer, saying that the computer knows where the ball moves, but we do not. She thinks that, in general, the white ball knows by itself where to move, as if somebody, on the computer, magically controls it. She also said that we could know where it is moving if we made rules that could control the movement. So, an idea of controlling the movement was for her to build some rules, and she realised that by having no rules about the ball's movement it was not possible for her to know where it went.

Tom (7 year-old boy) also expressed the connection between the arbitrary movement of the ball and randomness. He described how the white balls selected a ball to touch.

T: I don't know. They only have a speed. They don't do anything else. **They don't**

know. They just move and touch them randomly.

For Tom, the white balls are uncontrollable objects that are moving in a 'random' way. He argues that he does not know where the balls were moving and the balls do not know where to move either. It seems that this make him think of randomness to describe in the movement, so there was nothing 'hidden' in their movement that he had to find out. Tom looked and found the rules of the balls, discovering that the white balls had only a rule to move with a certain speed and did not 'obey' any a rule about how to move. This was what made him accept that a random behaviour was taking place in the game that could not be controlled by him. Irene's (7 6/12 year-old girl) reaction when she started the game was to describe the white ball's movement and then she tried to find the rules behind it. Initially she said 'It is just moving around without doing anything'. Her focus at the beginning was on how the white ball moved around and when she realised that she could do nothing about the white ball, unless she created a rule for the ball; she started thinking of how to place the coloured balls.

Anthony (5 10/12 year-old boy) connected arbitrary movement with chance:

Anthony: No, we do not know which ball will touch. It (the white ball) moved there and there and there...it selected a ball by chance. We didn't know where it would go. It picked a ball by itself and scored a point.

Researcher: Have you tried to **control** which ball to touch?

A: A little bit...we changed it a little...**adding balls inside** (red and blue balls)...

Anthony described the white ball as selecting a ball to hit by chance. He seems to understand the idea of chance in the game, and he also seems to believe that dealing with probabilities is an attempt to control, 'a little bit', the arbitrary movement of the white ball. It is arguable then that probability for Anthony is an attempt to control randomness. There is a change here in Anthony's approach to the game: he finds that he is unable to control the actual (random) movement of the ball, and so he tries to control the outcomes of the game by manipulating the coloured balls and thus constructing the events inside the lottery machine.

Similarly, Lucy expressed this feeling of controlling the randomness in sample space when she constructed a 'random controlled' situation of fairness. As she said 'I **made** the white ball move in order now they are going to have equal numbers'. Paul (6 10/12 year-old boy) also expressed this 'uncontrollable' idea about the white ball. He also decided to control the fair global outcome of the movement in a different way, attempting to make all the balls bounce around, collecting points.

Paul: Ok now! Ah! I know what to do!

He changes the balls and the speed again.

Researcher: The blues have more points than the reds.

Paul: Do you know what to do? We can take out all the white balls and give speed to the red and blue balls…when the blue touches a blue or a red touches a red ball, it gets two points otherwise it gets one point.

Paul gave movement to the red and blue balls and decided to change the rules under which points were scored and the mixture of the balls was made more complex. A similar attempt to change the system for a fair outcome based on the arbitrary movement of the white ball was George's (a 6 8/12 year-old boy) idea:

George: I have an idea. If the red touches the blue then it will be better. I will make the red smaller in order to move and touch the blue ball and then they will get the same points. Wow… It is a good idea! Bang! I will do something else. I will make these small and start the game.

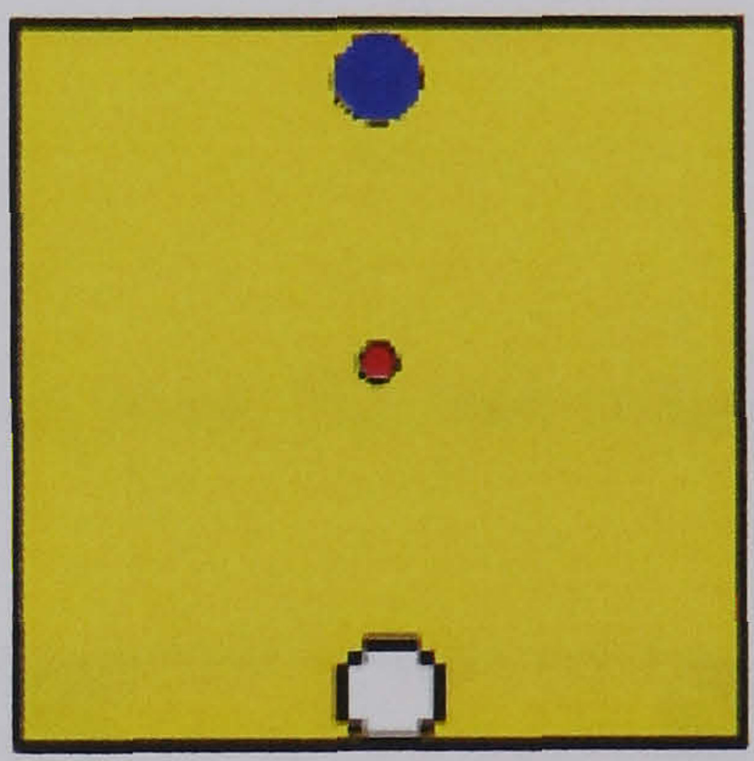

Figure 6.8: George's idea of changing the mechanism of the lottery machine

Again, George wanted to avoid the 'uncontrollable' movement of the white ball, moving around and collecting points. So, he decided to make the red ball move around so that it would touch the blue ball to get one point for the reds and one point for the blues, in order that the game would be fair.

This section has considered children's focus on the short term and long-term movement of the white ball. In the following section, I present cases where children's constructions focused on the *position* of the white ball in the lottery machine.

6.3 The position of the bouncing ball and the children's construction of sample spaces

The children's thinking about the spatial arrangement of the lottery machine can be categorised in two broad classes concerning how children thought about the position of the white ball. The analysis of the data presented here was based on code D2.2.2: Change the starting place of the white ball (see in section 4.3.5), which refers to children's expressions on the place of the white ball. The analysis in this section divided children's expressions on two categories: a. when the position of the bouncing ball matters (14 out of 23 expressed this idea at the beginning of the game), and b. where the position of the bouncing ball does not matter (22 out of 23 expressed this idea at the end of the game).

6.3.1 The position of the bouncing ball matters

In Rachel's case, it seemed the position of the white ball played a role in her fair and unfair constructions.

Rachel: The ball will move down and get the same scores. I think the ball will move like this and this and this.

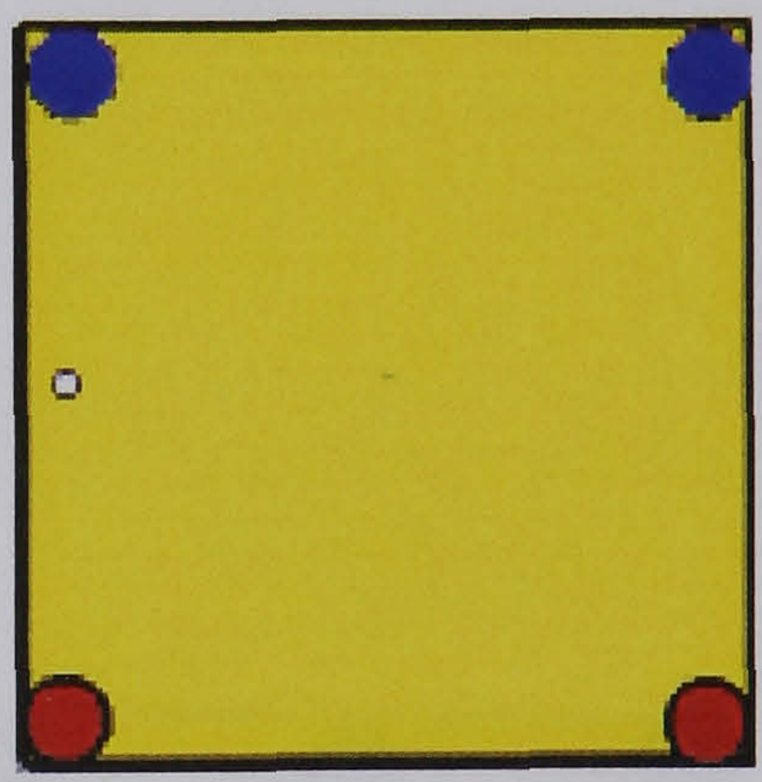

Figure 6.9: Rachel's first construction where the position of the bouncing ball matters

Researcher: How will the ball move?

Rachel: The ball, if it wants to touch the blue ball it will also touch the red ball as well. Or, it can move here and here.

She starts the game. The white ball did not touch any coloured balls.

Rachel: Well, I can also do something else, but I have to stop and start the game again.

Researcher: That's ok!

Rachel: I need to copy this... Ok! That's it.

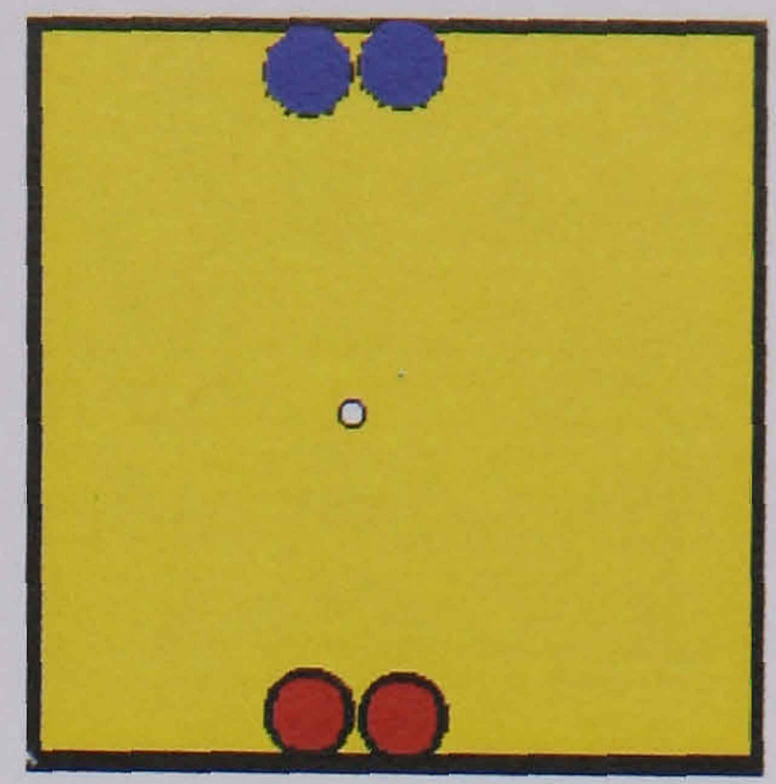

Figure 6.10: Rachel's second construction where the position of the ball matters

Researcher: What will we get now?

Rachel: Equal scores. The poor space kid will be there, getting rest. Let me start it.

She starts the game.

Rachel: D you know something? If I put a ball like this then the white ball will move like this and we get equal scores. Look! 25-27! I will try this idea now.

She stops the game.

Rachel: I will do something else to the balls. I will get the magic wand and copy one blue here and one red. We will have definitely equal scores.

Researcher: Why are you so sure?

Rachel: I don't know, but you will see. That's it.

She starts the game.

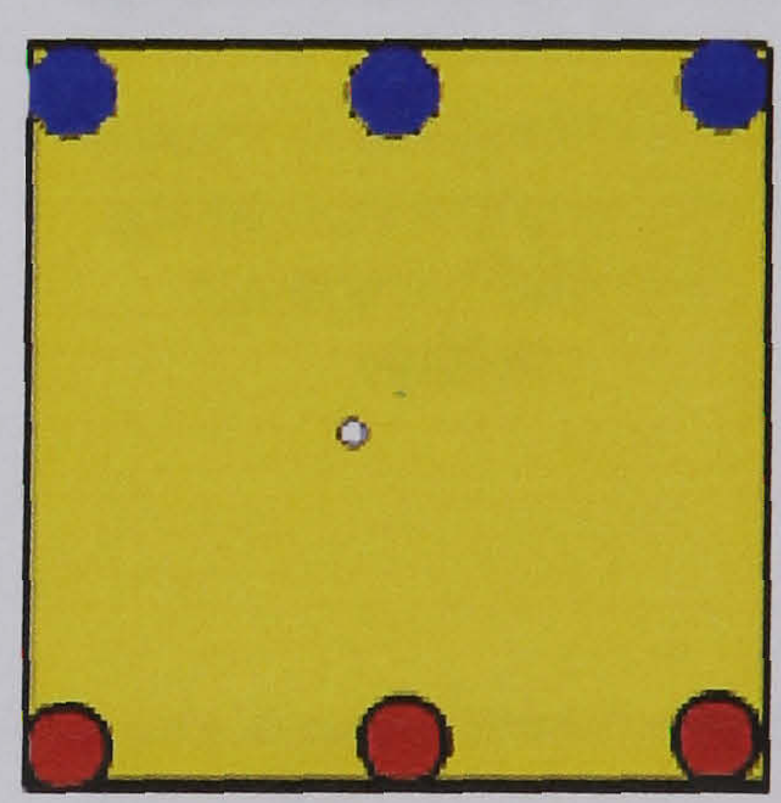

Figure 6.11: Rachel's final construction for a fair sample space where the position of the bouncing ball matters

In this episode, Rachel made three attempts to construct fair spatial representations. Actually, the last one did not substantially change from the first representation. The only issue for Rachel was the placement of the white ball. In her first construction, the white ball was in a symmetrical position between the coloured balls, but it could not bounce, as it did not touch any ball. So, instead of changing the position of the white ball, Rachel preferred to change the place of the coloured balls and, eventually, to introduce more of them in the lottery machine. She did not want to destroy the symmetrical position of the white ball, although, in these constructions, the position of the white ball would not affect the probability of an event. However, Rachel's strategy suggests that she wanted to place the white ball in the middle in order to construct a fair sample space and that she was concerned with the axis of symmetry. The white ball, in Rachel's fair constructions, always lay on a symmetrical axis of the coloured balls. Perhaps this 'invisible' axis was the concept behind the symmetrical arrangement of the coloured balls each time. This kind of strategy was very common in constructing fairness (see the following chapter, section 7.2).

In many of Rachel's asymmetrical constructions of unfairness the position of the white ball also plays a role, being placed in an asymmetrical configuration.

Rachel: I will do something ok! I will make them bigger! This is what I want to do.

Researcher: What are you doing?

Rachel: Whatever is in my mind! I don't know... Destroy them... and make these big enough. Oh...oh. I want to put these two together. Let's start the game.

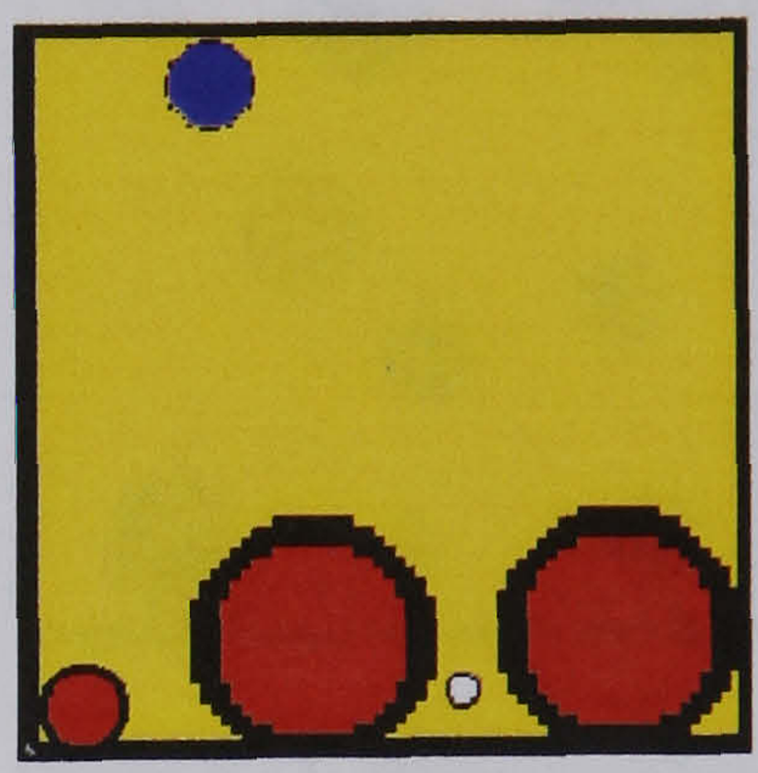

Figure 6.12: Rachel's first unfair construction where the place of the ball matters

She starts the game.

Researcher: Let's see if it is going to get any points...

Rachel: I am telling you it won't get any points....Only one is ok! Oh another one...oh...it stopped 2-19! That's good. I will do something else... You will see what happens!

She starts the game

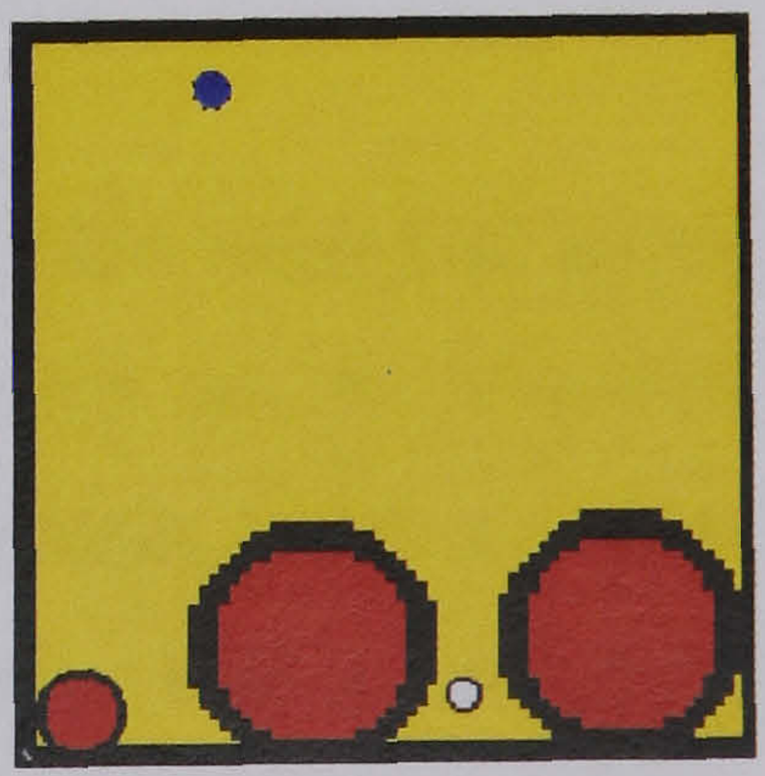

Figure 6.13: Rachel's second unfair construction where the position of the bouncing ball matters

Rachel: Nothing at all! No points… I made it so small we cannot even see it! Excellent! Did you see it? Yes!

The position of the white ball in these constructions seems to be influenced by her thinking about the short-term movement of the white ball. This is the reason that the result of her first trial was not as successful as she wanted. Eventually, she succeeded in getting the desired outcome, by making the blue ball smaller than before to decrease the probability of getting it. It can be said that here, where the balls were placed in asymmetrical form, the position of the white ball was not influenced by any 'invisible' axis, but rather it was connected with the short-term movement of the ball: Rachel wanted to trap, not very effectively, the white ball between the two red balls, as she wanted to increase the probability of the reds.

Another case concerning the position of the white ball in an unfair construction, influenced by short-term movements, is that of Nichol (7 8/12 year-old girl).

Nichol: Now the space kid will move down…5 reds and 5 blues, but the blue front balls are more, the white ball will move like this and touch the most. It is a bit mixed up, but not exactly.

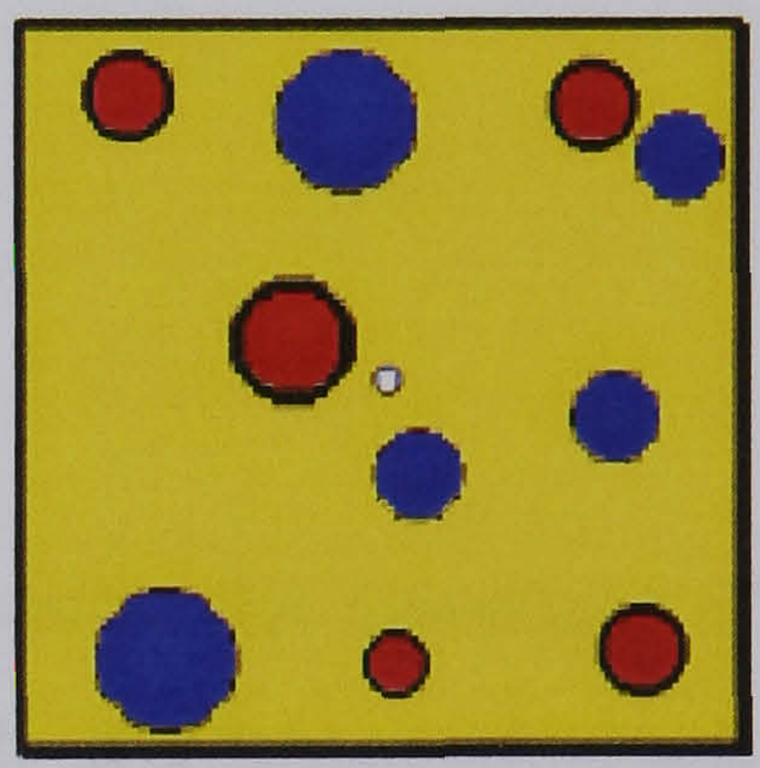

Figure 6.14: Nichol's unfair construction where the position of the bouncing ball matters

She starts the game.

Nichol based her decision on the short-term movement of the white ball and because she wanted the blue to win she placed more blue balls near the white ball.

On the other hand, there were also some cases where the position of the white ball played an important role in the spatial construction of unfairness focused on the long-term movement of the white ball, where blue or red balls trap it. For example, in Lucy's unfair spatial arrangement, for blues to win, the place of the white ball makes a difference in her construction.

Lucy: I will put the red ball on the edges. **I will do a wall** with the balls and I put the white ball on the blue balls.

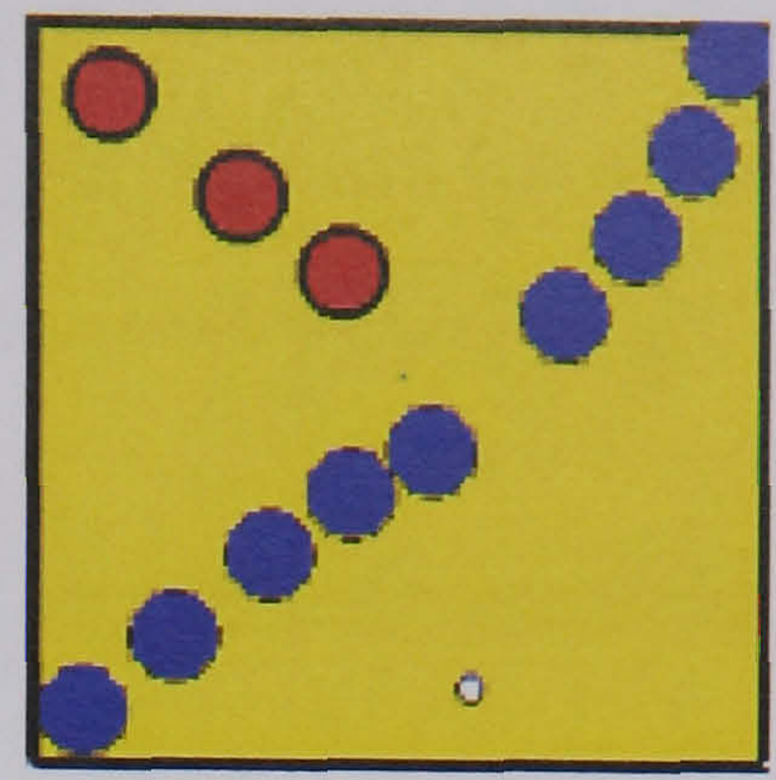

Figure 6.15: Lucy's unfair construction where the position of the bouncing ball matters

Lucy here tried to trap the white ball and make the space kid get easily to the blue planet. It can be concluded here that Lucy wanted to set up a separate sample space of blues. She made a 'blue wall', so that the white ball was trapped and it was very difficult to touch any red balls. Not only did the number of the balls essentially increased the probability of getting blue, but also was the spatial arrangement of the balls and the placement of the white ball. The following section describes some of the cases where the position of the bouncing ball did not matter.

6.3.2 The position of the bouncing ball does not matter

Almost all the children (22 out of 23), after playing the game and seeing the movement of the ball in the long term decided that the position of the white ball in a fair environment does not matter. For example, in John's (6 10/12 year-old boy) case of constructing a fair environment with the same number of balls in two separate 'teams', he placed the white ball in a place, so as to influence the long-term movement of the white ball.

John: Let me copy some…. 1,2…5 blue and I will copy 5 here. 1,2,..5 red balls!

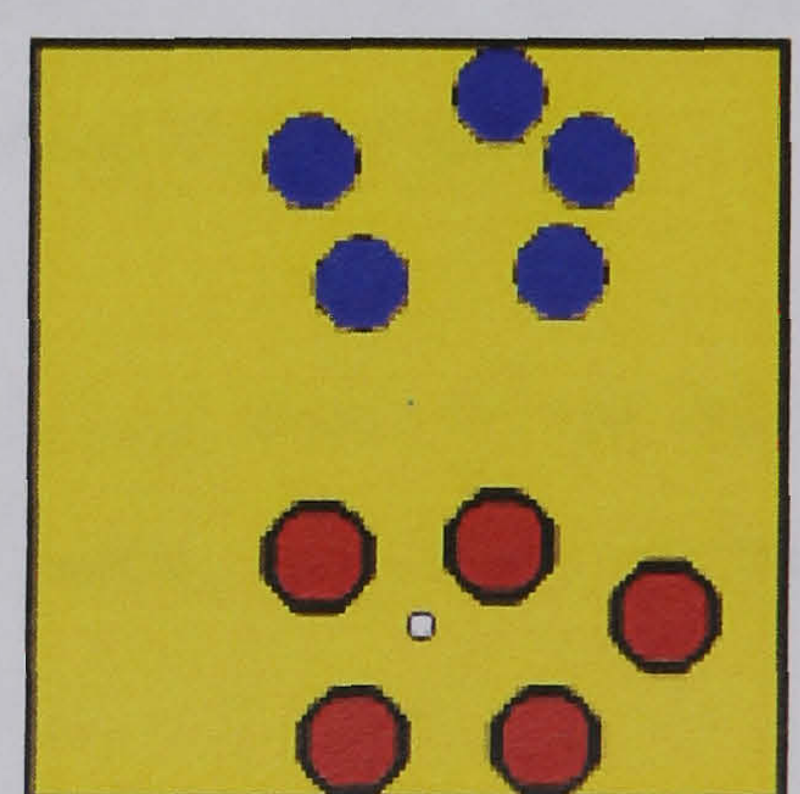

Figure 6.16: John's construction where the position of the ball does not matter

John constructed two similar spatial arrangements of the two colours, indicating in a way the equality of these two arrangements. I expected John to place the white ball in the middle, as 14 out of 23 children did at the beginning, to show the axis of symmetry between the two teams. But John was more influenced by the long-term movement of the ball- he thought that the place of the white ball would not play a role for his desired result.

The same happened with Cathy (7 6/12 year-old girl), who she also did not place the white ball in the middle of her fair construction.

Cathy: I will destroy some balls. Now, I have three reds and two blues…I will destroy another ball from the reds. Let me see how I will arrange them… Hm…

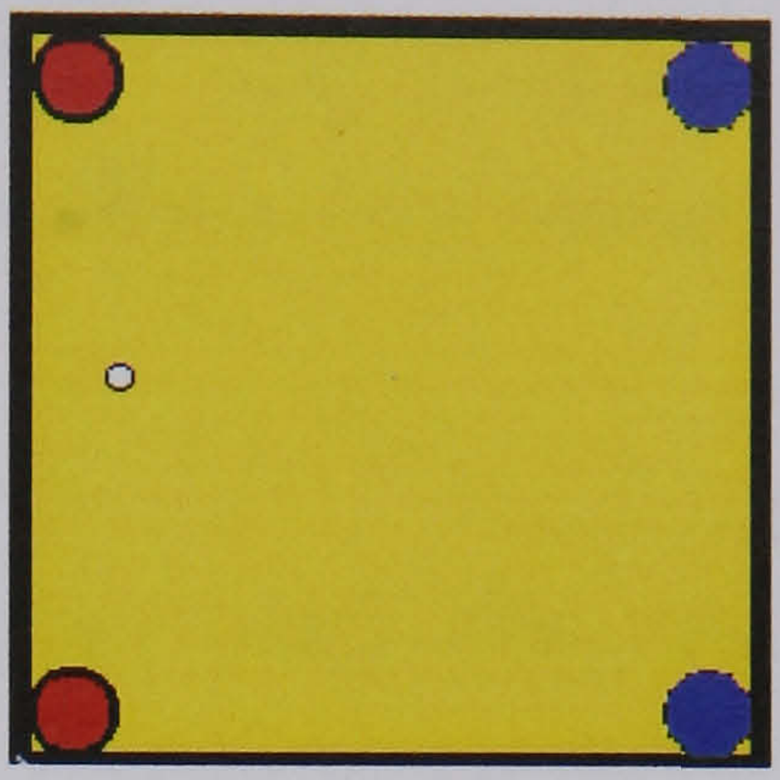

Figure 6.17: Cathy's construction where the position of the ball does not matter

Researcher: How did you arrange them?

C: I placed them in a way to move up and down.

Cathy describes her decision here without worrying where she had placed the white ball. As Simon (7 10/12 year-old boy) said in a similar construction 'It doesn't matter where you placed the white ball. It will move in different ways. Diagonal, zic-zac…somehow…it is bizarre… Wherever you placed it, it will slip over and touch the balls…' This 'bizarre' movement made 22/23 children to think at the end that it does not matter where the white ball is placed in a fair sample space, as it will 'slip over' and get the other balls. The children realised the limitation of being able to find the exact movement of the ball. It can be said that this realisation was the point where children started to think about the existence of randomness, described by Simon as a 'bizarre' happening in their game.

The final section will describe how children connected the local events, either by the rules or by the long-term movement of the white ball, with the global events, the scorers or the movement of the space kid.

6.4 The children's connection of local and global events

The connection between the movement of the white ball, getting points and making the space kid move up and down seemed to be for all children an easy target to achieve, as they were familiar with making rules and connecting one object with the another in the game (see section 4.3.2). Of course, this was an intention of the design, for children to have both local and global events (see Chapter Three, section 3.4.1) on the screen and to use this visualisation to make connections. Some evidence of using the visualisation comes from Simon's reaction when he started the game:

Researcher: Would you like to start the game and tell me what happens?

Simon: This little ball is moving and when it touches the balls it moves up and down and this triangle (the space kid) moves up and down and opens its wings.

R: When does it move up?

S: It moves...Ah! We have these counters that count how many times the white ball touched the blue and the red balls.

For Simon it was easy just to watch the game and describe what was happening. On the other hand, Mathew (7 year-old boy) wanted to be surer of how the game was working and he examined the rules of each object.

Mathew: I cannot imagine the rules, but... I think the space kid will move up when it (the white ball) touches the red and when it touches the blue ball to go down. Can I find out? Let me see... 'When it gets the blue envelope (iconic for message) to move down, when it gets the red envelope to move up'.

Researcher: Where are these envelopes coming from?

M: From the counters. Let me see... 'When the game starts it counts from zero and when it gets a red envelope it adds one point'.... Ah! I know. The counter gives it to the space kid and moves up... Ah... they get the envelope. Let me see these rules for a minute! It shows the same thing 'when the game starts is zero and when it gets the blue envelope it adds one'.

R: Who is sending these envelopes?

M: Let me see whether the blue has a rule... It does! 'When I touch something I push it back and I send a blue envelope'....ah... it is sending the blue message...when I touch the white ball I push it back and get points. Let me see the red one...it is the same 'when I touch the white ball I push it back and send a red message to the space kid and to the red counter'. Ah! These are sending the messages when the white ball touches them!

Mathew here developed an understanding of the game and the connection between sample space and global outcomes by using the rules of the game. But, how did these connections affect children's probabilistic decisions? The following paragraphs illustrate some extracts of how children made their probabilistic decisions.

Jane first made her fair construction and then she used the global effect of it to decide whether her sample space was fair or not.

Jane: I can make the red bigger as well.

Researcher: And then?

J: The scorers will be equal.

She takes the star tool and she makes the red ball bigger.

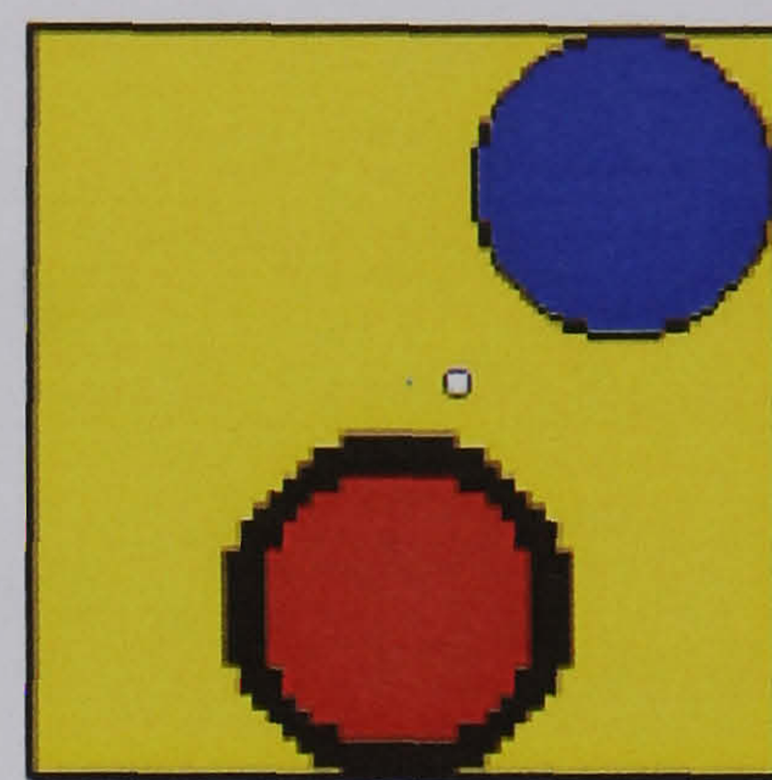

Figure 6.18: Jane's fair construction based on the global effect

J: I think the red will win.

R: Why is that?

J: I think I made it a little bigger than the other... We can open the game and if the scorers are the same that means that they have the same size, otherwise the one is bigger than the other.

R: What about the space kid?

J: If it is as now that means our balls have the same size...

Fairness in Jane's sample space was judged from the global outcome of the game. Jane had in her mind that in order for her environment to be fair the balls should have the same size. The scorers and the space kid shaped her decision as to whether these two balls had the same size or the one was bigger than the other.

Anne (6 10/12 year-old girl) used the global outcome of the game (the sounds) to show that her fair construction was working.

Anne: It didn't work! I will do something else now...I'm making it a little fairer. I want the balls a bit more nearly.

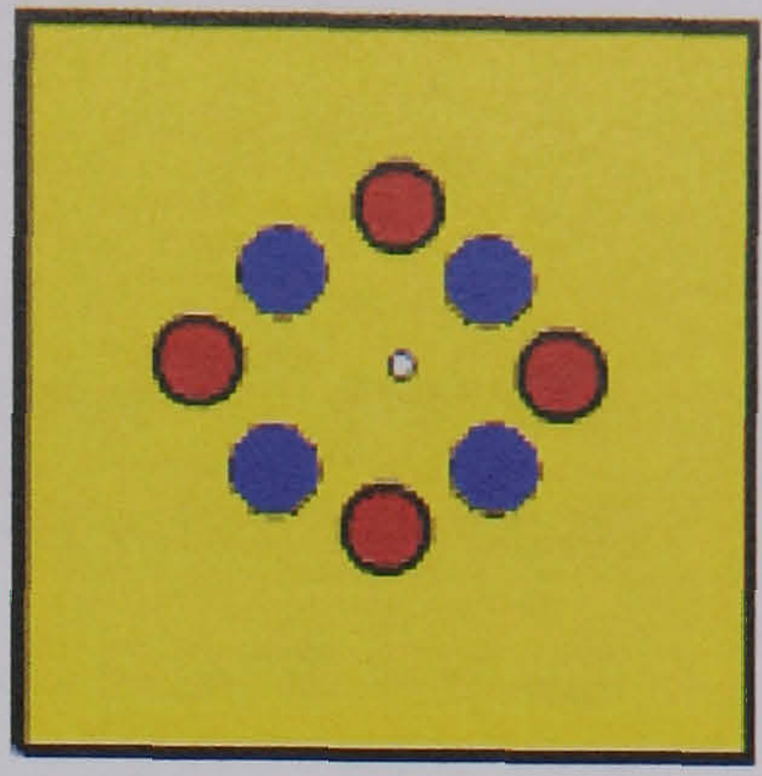

Figure 6.19: Anne's construction based on the global outcomes

Researcher:	Do you think now it's going to work?
A:	Let's see…
	She switches the game on.
	It's working!! It's near the yellow line! It should go 'boo', booing', 'boo'…
	She uses the sound that the space kid makes when it moves up and down.
R:	Why do you think is working now?
A:	Eh...they (the points on the scores) are near!
	She switches off the game and changes the sounds. I'm making the same sound!
R:	Do you want to have the same sounds?
A:	Yes…because if you are at the other side of the room, you won't know which ball it was touching. What I want you to do is to go to the other side and try it out!
R:	Ok…Shall I just shut my eyes?
A:	Ok! Ready?

Anne here expressed her belief that her construction is fair by changing the sounds of the space kid and making them the same. Perhaps what she wanted to show was that the total of the two different sounds would be the same, as the space kid was going to stay near the yellow line and the global outcome of her fair game would be neutral. Also she asked me to go to the other side of the room to tell her which balls the white ball was touching. She might have wanted to show that in the long term for a fair probabilistic game I would not be able to distinguish which event on a short term is winning.

6.5 Summary of Chapter Six and provisional findings

This chapter described how the children in this study linked the local and global events of the lottery game. The continuous 2-dimensional movement in the lottery machine appeared to provide a rationale for children to connect the short-term behaviour of the game with the long-term outcome. It was also a way for children to accept that the movement of the ball was arbitrary, and furthermore to accept the existence of randomness in the game. The analysis of data has shown that all the twenty-three children paid attention to the position of the bouncing ball in the construction of their sample space when they were focusing on the short-term behaviour. The starting point of the white ball was an invisible axis of their construction. The episodes considered here lead to the following provisional findings, some analytical points that emerge from the local analysis of the data-at-hand, but which will be fleshed out in more detail in the final chapter when it can be looked at more globally in relation to other data and findings from the theoretical framework of the thesis outlined in earlier chapters.

Provisional finding 6.1: All the children tried to find patterns to 'explain' the movement of the white ball. These patterns were based on the path traced by the white ball in order to 'achieve' a hit. However, because of the continuous movement of the ball it quickly became impossible for them to predict. It was this impossibility that led to the interpretation of the ball's movement as 'arbitrary'. This arbitrary movement was the 'urging force' for children to express their ideas for randomness.

Provisional finding 6.2: It might have been expected from the children's age that constructing a meaning for randomness would be extremely difficult. A reason for that would be that children of this age tend to focus on controlling the 'thing' that delivers randomness. However, the episodes above indicate that in the dynamic/spatial medium of expression provided by the lottery game (placement of balls, changes of size and number) children did construct meanings for randomness in the sense that they realised the need to control the outcome without controlling the random movement.

The next chapter describes how the children experienced fairness and unfairness in their lottery games. These episodes mainly occurred when children sought to find ways to control the outcomes of the game, *after* having connected local and global events, and having accepted the existence of randomness.

CHAPTER SEVEN

The Construction of Fairness and Unfairness

' I like to have fairness...that they had the same score... Because they didn't get one more than the other. They got almost the same numbers, may be a little more. It is fair both of them to win...they are not going to cry and fight'.

(Anthony, 5 and 10 months year-old boy)

7.1 Overview

Like the previous chapter, this chapter is also part of the Phase 2, learning investigation phase, and the data analysed in this chapter came from the final iteration (iteration 3). The final iteration involved children working with the 'Space Kid' and its focus was on describing and analysing how the game mediated the children's expression of chance events. This chapter will be divided into three parts. The first part deals with *fairness*, and the strategies that children employed in order to construct a fair environment. These strategies can be seen as falling into two categories, symmetrical and asymmetrical. The second part describes how children constructed *unfairness* in their games, illustrating how children built unfair situations and how they handled certain and impossible events. The final part of this chapter gives a summary of the chapter and some further analysis and provisional findings of the data that have been described in the previous two parts.

7.2 Fairness

As the game consisted of two 'teams', the blue and the red team, all the children expressed the idea to create a sample space where these two teams would get equal points. As Anthony described, fairness in a game is something that 'makes you not cry', so it is an important factor for playing a game. John (6 10/12 year-old boy) also explained that fairness was something that was 'good' and 'right' to have in a game.

Researcher: Ok...Let me copy another blue ball...(there is one red ball and one blue ball in the lottery machine).

John: Why? Don't!

R: Why? We can do that with the magic wand.

J: No! You cannot do that! It is not right, it is not fair!

This children's wish to play a fair game was a strong motivation for them to construct a fair random sample space. After Chris' (7 8/12 year-old boy) first attempt to construct a fair sample space, he stated that fairness was not an easy thing to construct in a probabilistic game.

Chris: Oh...it moves upwards... I don't know what to do. I will try something else...You know, it is too difficult to do it!

Researcher: Why is it too difficult?

C: We don't know how this is moving... I will try something else.

The uncontrolled movement of the white ball made fairness in the game a difficult condition to be achieved, but an interesting one to be constructed. The analysis of the data presented here was mainly based on Code D3.1: Construction of fairness (see Chapter Four, section 4.3.5), which referred on the children's expressions of constructing fairness. The following section describes children's strategies for constructing a fair environment; these can be seen as falling into two categories, symmetrical spatial representations and asymmetrical spatial representations.

7.3 Strategies for the construction of fairness

The strategies that children developed for constructing a fair environment can be generally divided into two categories with several sub-categories:

a. Symmetrical spatial representations: symmetrical balls, symmetrical groups of balls, making patterns, making circles
b. Asymmetrical spatial representations: equal number and size of two balls, mixed up balls: equal number and size of balls, different size and number of balls

The analysis of the data presented on the first category (symmetrical spatial representations) was mainly based on Codes D3.1.1: Moving and changing the number of the elements of sample space with its sub codes D3.1.1.1: Symmetrical teams, D3.1.1.2: Making a pattern, D3.2.2.3: Making circles. The analysis of the data presented on the second category (asymmetrical spatial representations) was mainly based on Codes D3.1.2: Changing the size of the elements of sample space, D3.1.3: Changing the arrangement of the balls and D3.2: Construction of unfairness (see Chapter Four, section 4.3.5).

7.3.1 Symmetry of placement to represent fairness

Twenty-two out of the twenty-three children tried to achieve a fair game by placing the balls symmetrically. Their representations of a fair sample space were associated strongly with symmetry, expressed by: a. constructing symmetrical balls, b. constructing symmetrical teams, c. making patterns and d. making circles

7.3.1.1 Representation of fairness with symmetrical balls

Children used 'symmetrical balls' for construction of a sample space where the two colours were to have the same probability of an event. The characteristic of this strategy is that the symmetrical *position* of the balls matter. Children who constructed an environment by using this strategy were very concerned about where each ball was placed, as Mathew (a 6 year-old boy) described for his construction

Mathew: The trick is to put them separately, near the white ball and get the same points, near to each other in the middle. Oh… It gets the red more times…. It needs another 4 points to be equal… Ops… Come on…come on….again…Equal!

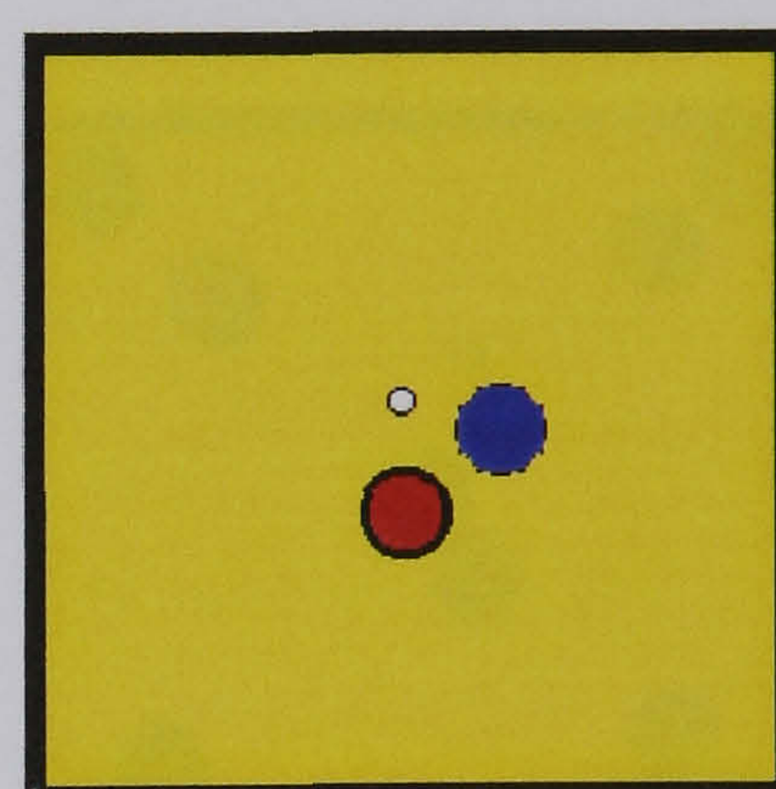

Figure 7.1: Mathew's fair symmetrical construction

Mathew found that the 'trick' of fairness in the sample space was to have symmetry between the two balls. He connected fairness with the idea of balance. The blue ball was placed near the red one and the white ball was placed in the middle, showing the symmetrical axis. Tom (7 year-old boy) made a similar construction. He also placed the two balls symmetrically.

Tom: Let me check the rules again… Ok! I will change the size of this ball and I will make this to be the same. I will put them like this. I will make the white balls bigger. Ok. I will move them like this…

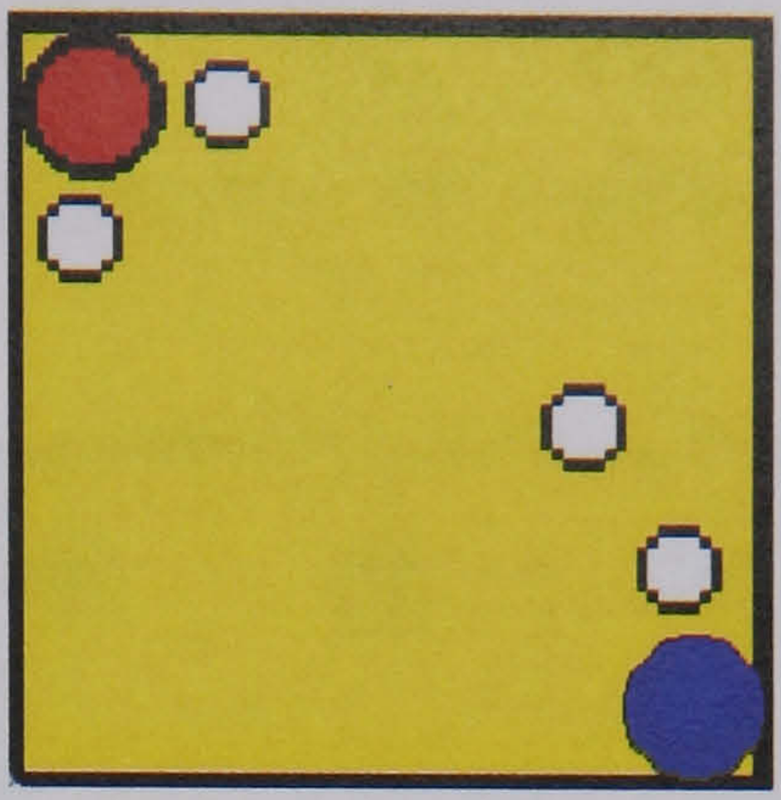

Figure 7.2: Tom's fair symmetrical construction

R: How did you place them?

T: One to the one side and the other to the other side.

Tom constructed fairness in the game by having balls on opposite sides of sample space, equidistant from the centre. Tom also made the balls bigger than before, so that it 'is sure that we are going to get 100 points quickly'. (This quantitative aspect of randomness will be analysed in Chapter Eight.)

Symmetry between the balls was also expressed in Simon's (7 10/12 year-old boy) construction. He placed his construction by placing each blue ball symmetrically with a red one.

R: What are you trying to do?

Simon: Something alike a cross. The (white) ball will touch each ball…

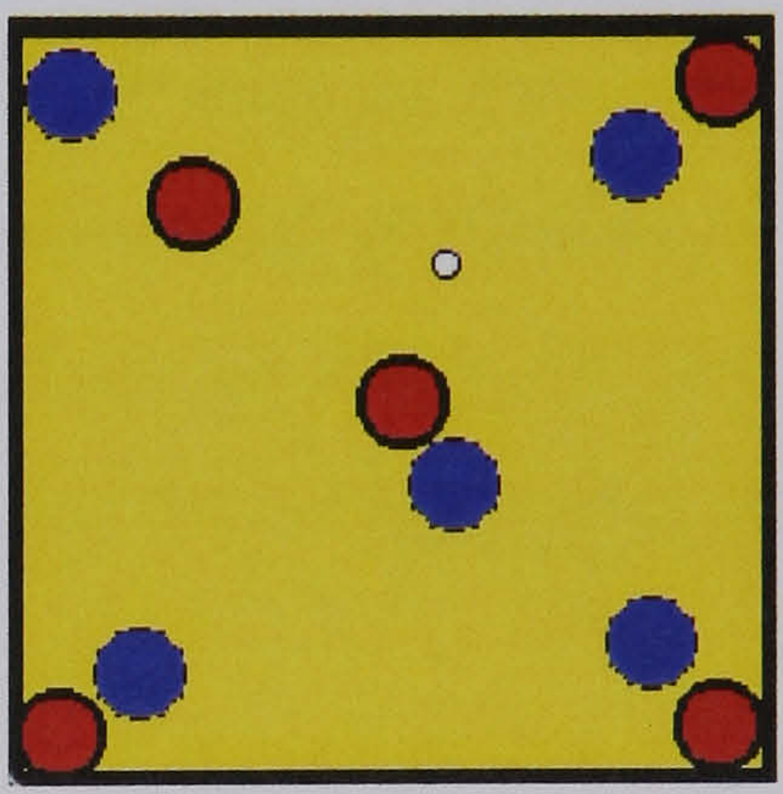

Figure 7.3: Simon's fair symmetrical construction

Simon's symmetrical construction needs to have each blue and red ball in a particular place and each pair symmetrically placed with another. His construction was made in a way such that the collisions with either colour had the same probability of occurring. This was achieved by placing the balls symmetrically, so that when the white ball touches one colour it should touch the other colour as well. This construction might be seen as a construction based on thinking about individual events in the lottery machine.

Getting a big score quickly was also what made Karen (7 3/12 year-old girl) put more balls into her fair construction, placing the balls again in a symmetrical position. She worked with the balls both individually and as groups.

Karen: I put here one red and one blue and then three red balls and three blue balls.

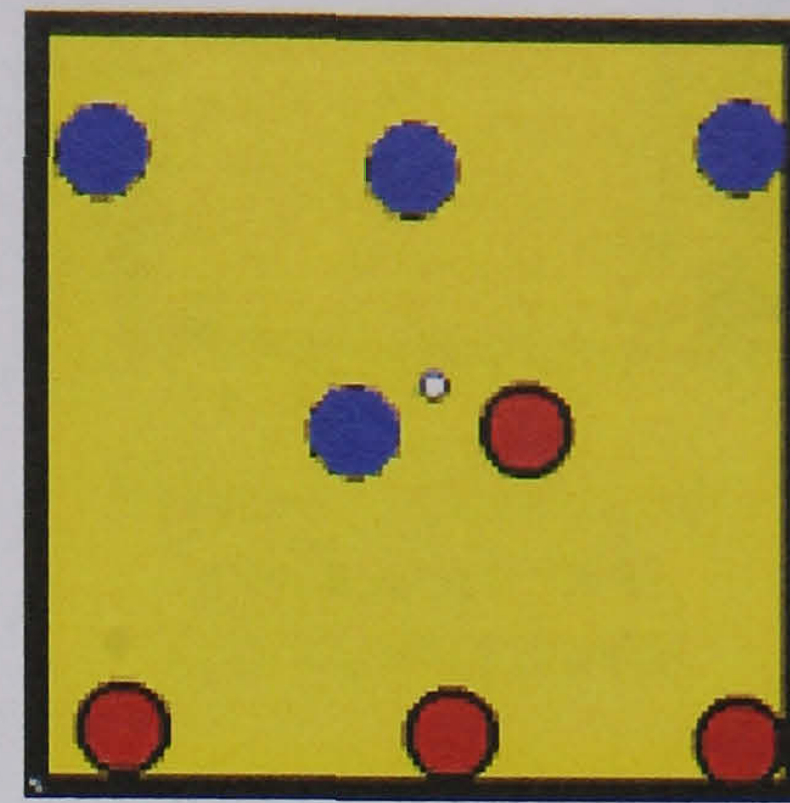

Figure 7.4: Karen's fair symmetrical construction

Karen's description shows the way that she thought of placing the balls inside the sample space. She first described the place of the two single balls, which put in the middle with the white ball between them, and then she described the other balls of each colour, referred to them as two groups of three balls placed in symmetrical positions.

There were eighteen out of the twenty-three children who represented fairness with a symmetrical ball construction. The children who made this construction were very concerned about where each ball was placed. The blue ball was placed symmetrically to the red one and many times the white ball was placed in the middle, showing an 'invisible' symmetrical axis. It seems that the children who started with a symmetrical placement of individual balls developed their representation to symmetrical groups of balls. In fact, working with symmetrical groups was another common strategy for the construction of fairness, as described in the next sub-section.

7.3.1.2 Representation of fairness with symmetrical groups of balls

In this strategy children did not seem to care so much where each individual ball was placed. They were concerned with the positioning of groups of balls, seeing the behaviour of the sample space in the long term. This is obvious in Brian's construction (6 6/12 year-old boy), who not only separated the two colours, but also made this clear by putting lines between the two teams, defining new rules for these new objects.

Brian: I want to make a line… I will blow this ball… I know, I will do something.

He separates the two colours and he constructs a new object

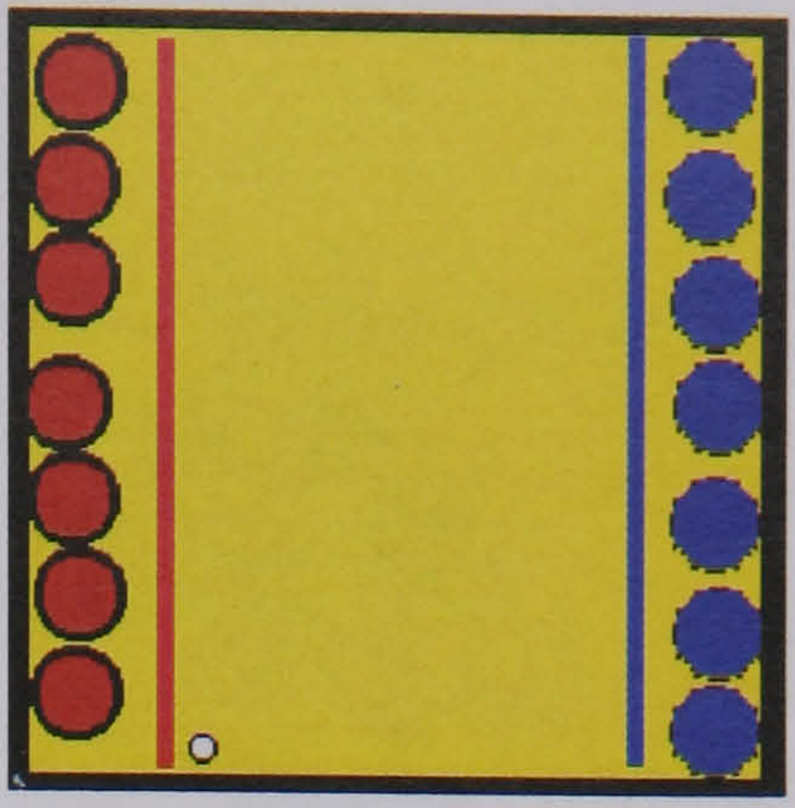

Figure 7.5: Brian's fair construction with symmetrical groups

R: What do you want to do?

B: I need a rule for this object. 'When the ball touches me, I take all the messages from all the red balls and I give them to the red scorer'. I need the same to happen to the blue balls.

Brian made it obvious in this construction that he treated the coloured balls as separate teams, putting a line between them and distinguishing them. As his rules were stating, he wanted the balls to give messages to the two lines to add a point to the scorers. Brian used this approach to prevent any possibility for the white ball to remain in the gaps between the balls, a possibility that he had seen earlier.

The separation of the balls into two groups was also expressed by Karen's final construction of fairness.

Karen: I just thought of it! Look! This will be my last try! That's it! Two lines.

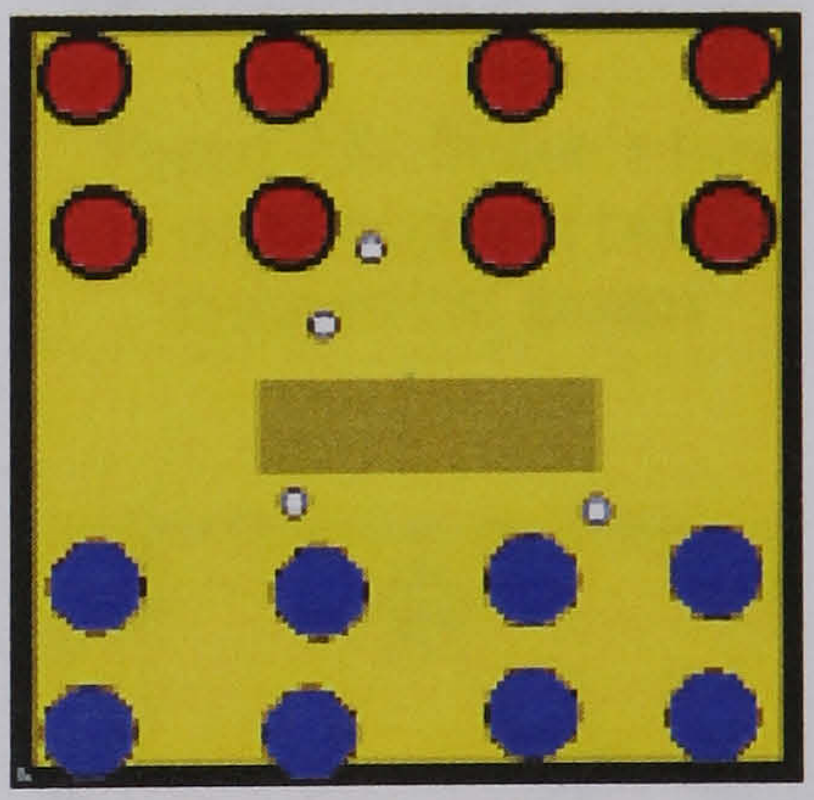

Figure 7.6: Karen's separation of the two groups

Karen separated the two groups symmetrically, putting a brick between them, which had the rule (see section 3.3.3) 'when I am touching any object I bounce it off'. This brick aimed to reflect off the white balls. Her actions suggest that she might have thought to have two separate sample spaces, one for each colour each with its own white ball inside so as to collect the same number of points. She was trying to have an equiprobable sample space by dividing it into two parts one for the one event and one for the other.

Jane also described the construction of two symmetrical groups:

Jane: I'll put near all the red balls and then all the blue ones. Then, I will take these two teams and I will put them near each other.

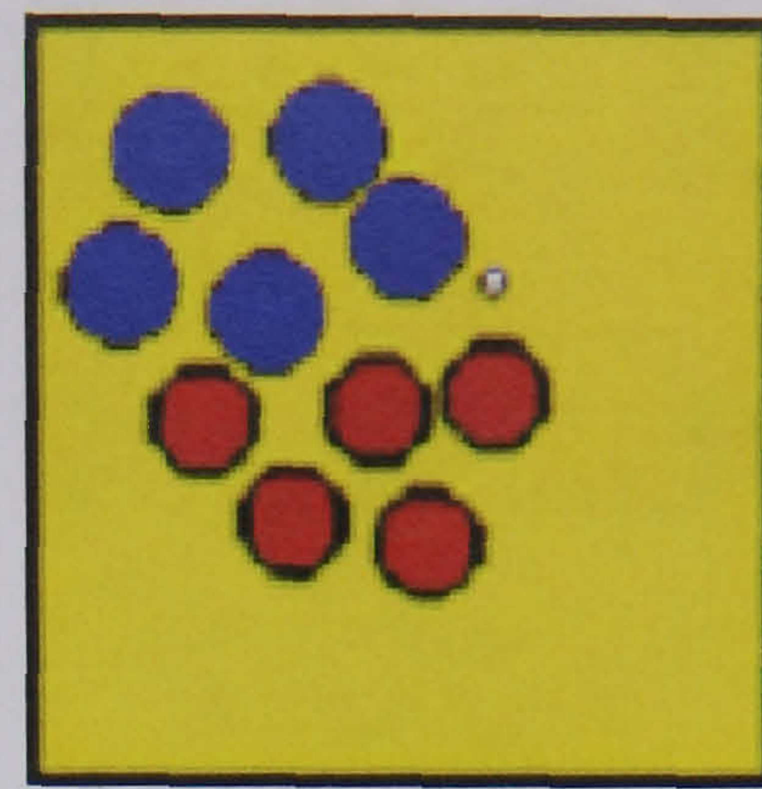

Figure 7.7: Jane's fair construction of symmetrical groups

She starts the game.

J: Oh! They are equal!

In Jane's idea, the two groups were placed near to each other and the white ball was placed in the middle.

Simon also worked with symmetrical teams:

Simon: ...Let me show you...This will go here and this one here...I think is ok! It will move up the same times that it will move down.

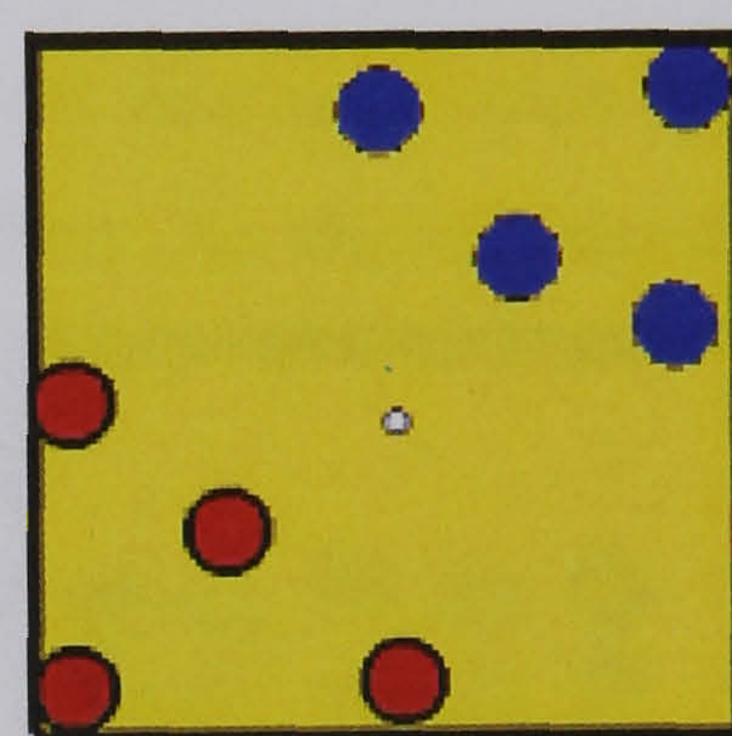

Figure 7.8: Simon's first construction of two symmetrical groups

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S: Ok! I can do it! That's ok! I added another blue and another red ball.

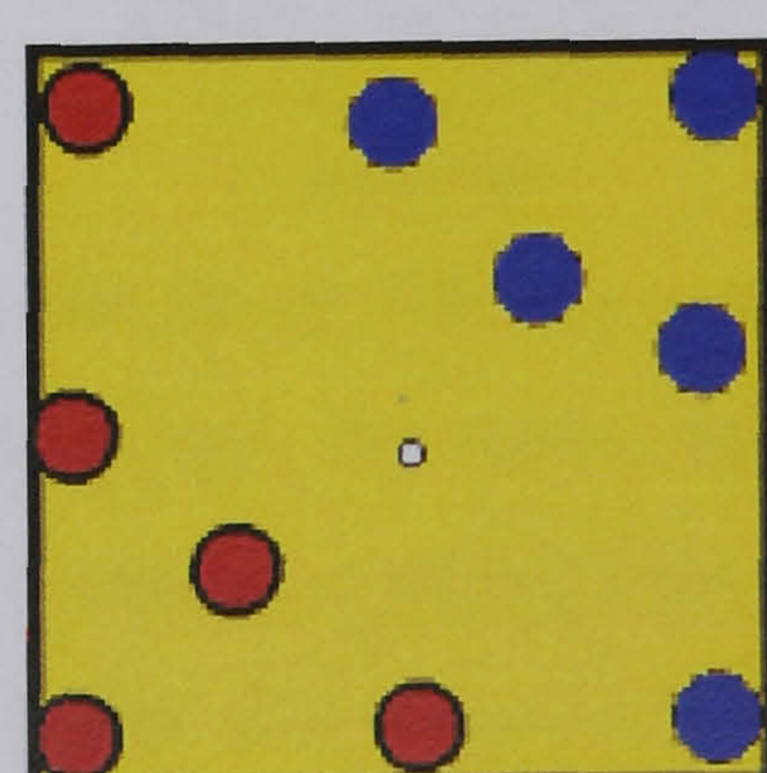

Figure 7.9: Simon's fair construction of symmetrical balls and groups

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S: You know something. I will add some more balls. I will put these balls together…to communicate (he laughs).

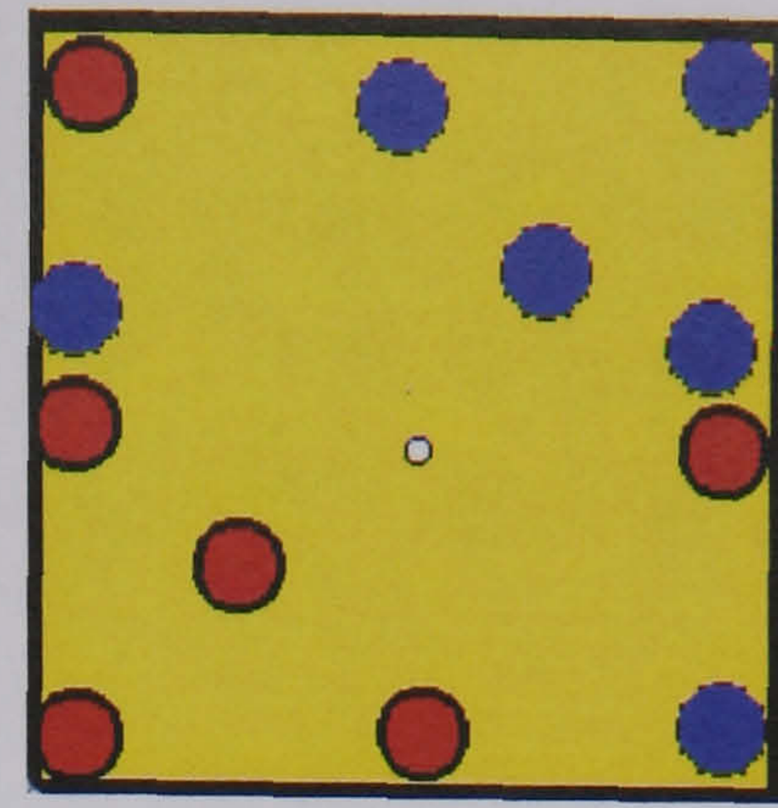


Figure 7.10: Simon's final fair symmetrical construction

R: What do you think will happen?

S: We are going to have a better result.

Simon first constructed a sample space with two symmetrical teams with the same number of balls, and then he continued his construction by adding two more balls. He placed them by having in mind an imaginary diagonal symmetry axis. He finished his construction by placing another two balls symmetrically, having now a vertical and horizontal symmetry axis.

A different approach was Fiona's (7 year-old girl) first construction of fairness, which also has to do with two symmetrical teams, but not with equality of numbers. As she explained:

Fiona: Our task is to make the blue score to get as many points as the red score. So, I put them like this. First of all, let's put this here. Number five. This to go there and another two changes. I think it will keep that on the line. I am trying to make this remain stay. Let's try it out.

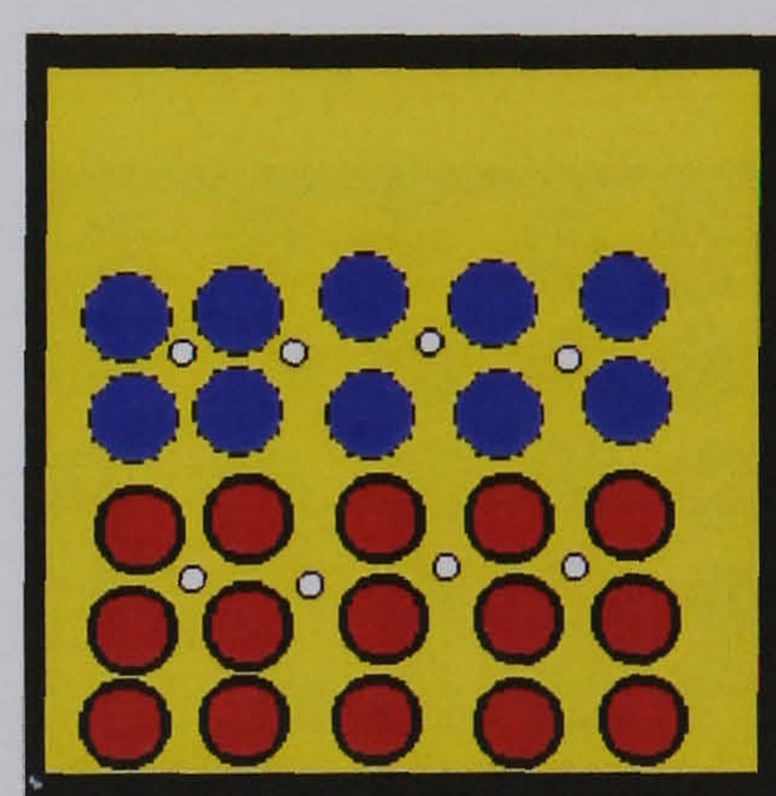


Figure 7.11: Fiona's symmetrical fair construction

As she explained, the reason behind this construction was to make the space kid stay near the yellow line. It seems from her sample space that she tried to contain the white ball between the blue and the red rows; she did not care about counting the number of the balls. Generally, twenty out of the twenty-three children represented fairness with symmetrical groups of balls. In this strategy children expressed an idea of having two separated sample spaces, one for each colour, with the same structure and working in parallel to collect the same number of points. Furthermore, Fiona's symmetrical fair spatial representation did not work. The blues got more points than the reds because the white balls were blocked in the upper part of the sample space. She also expressed her representation of fairness by separating the two events, implying two separate sample spaces. But, the way she constructed these two sample spaces was biased. This caused her to move to a new construction, by placing the balls in a patterned way. Her idea is described in the next subsection, which illustrates how children made patterns for constructing a fair sample space.

7.3.1.3 Representation of fairness with a pattern

In Fiona's use of pattern, she first constructed one line of balls and then repeated it in order to develop her pattern. As she described:

Fiona: I am making this on a line, so...if it goes like that it will stay around the line. Then, I have to repeat this…because I have all the balls like that…Here… So, it's like a pattern on a line.

Researcher: Will it help you?

F: I think so… So let's put this here… Now it will work! Yeah… If we put this here…. That's ok! It has to work!

R: Why does it have to work?

F: I put them all in a different place. That has to work!

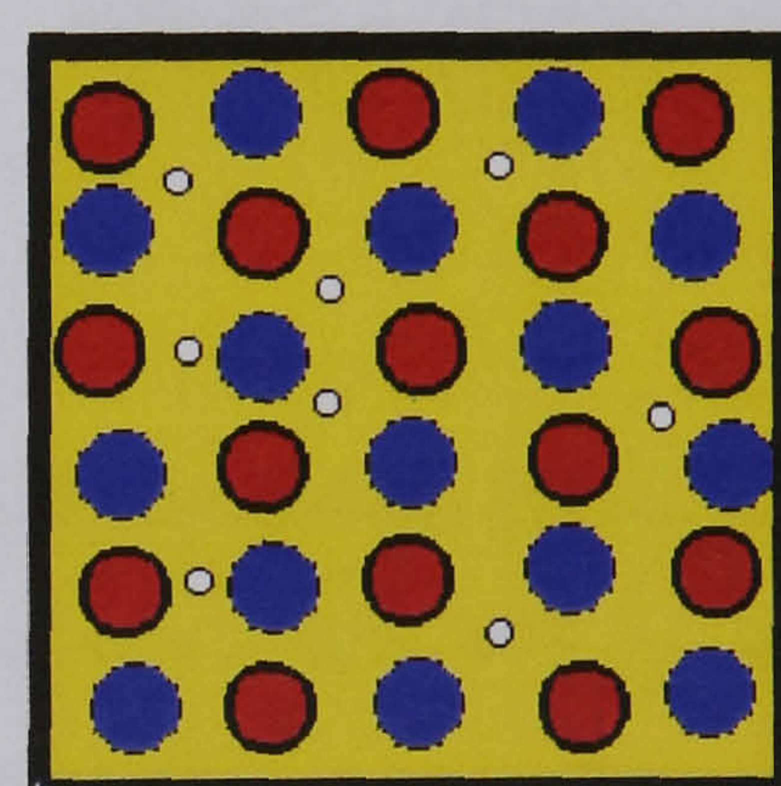


Figure 7.12: Fiona's patterned fair construction

Fiona was very positive that this construction would work. The idea was to create first the 'sample' line of the pattern and then to repeat it until the yellow square was filled. This solution for Fiona had to work, and her feeling that it is 'correct' was very strong. Patterns were also used as a way to have an equal number of red and blue balls, without counting. Lucy, in a similar construction, described how a pattern was working

Lucy: They are going to have equal numbers. It (the white ball) will move up, on the edges…the ball will get the same points. I will also copy another white ball to move quickly…They will become rows.

She starts the game.

Researcher: What happens? Where do the white balls go?

Lucy: It goes everywhere…around the balls. They have equal numbers now! I got one ball and another. I made a row and then another row and I made the white ball to move in a way and now they are going to have the same numbers.

The logic behind this was for one colour to be near the other, so that when the white ball was going to touch one colour it would touch the other as well.

Chris' (7 8/12 year-old boy) idea was a combination of symmetrical groups and making patterns.

R: What are you doing?

C: I am arranging one red, one blue, one red, one blue. I am building a wall!

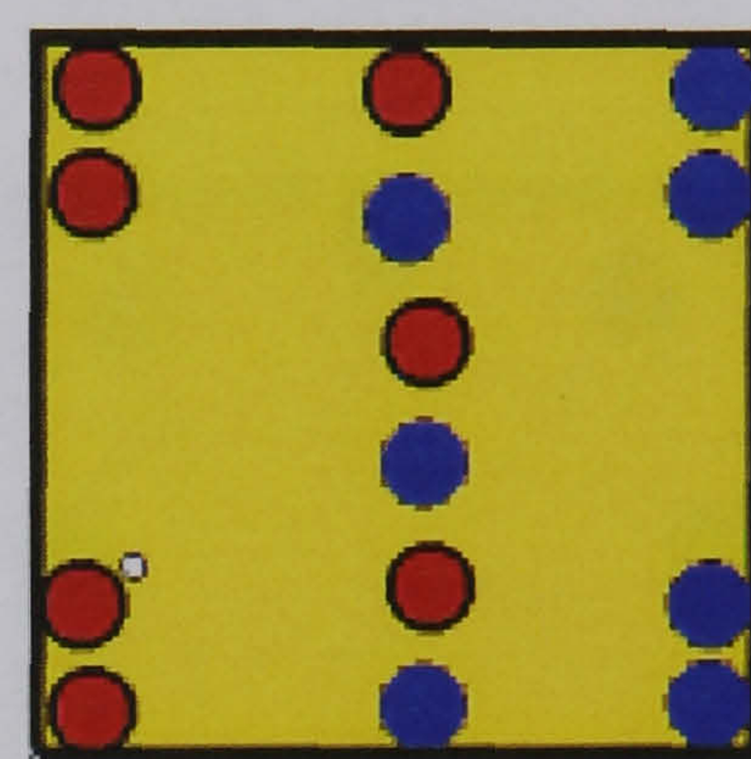


Figure 7.13: Chris' patterned and symmetrical groups fair construction

R: Ah! Why have you made a wall?

C: You will see!

Chris first made a sample space in which the two teams were spatially separated. He added a wall of alternately placed red and blue balls. He seemed to construct this wall as an expression of having two separate sample spaces. As is shown from this construction of fairness, a strategy does not always operate alone; children may combine several strategies in order to achieve the wanted result. A patterned spatial arrangement of the events was expressed by twelve children. The logic behind this was for the balls of one colour to be so near to balls of the other colour that if the white ball touched one colour it would touch the other as well. The next section refers to a symmetric circle representation of fairness.

7.3.1.4 Representation of fairness with circles

Another symmetrical strategy that eight out of twenty-three children used for the construction of fairness was to make circles and trap the moving balls inside them. In Cathy's (7 6/12 year-old girl) case, a pattern was developed into a circle.

Cathy: Let me see…ok! I will put it here… But, will it move where I want? … I will put these two there. Ok! I will put the white ball in the middle to get every ball. It is like a cross! Let's try it out!

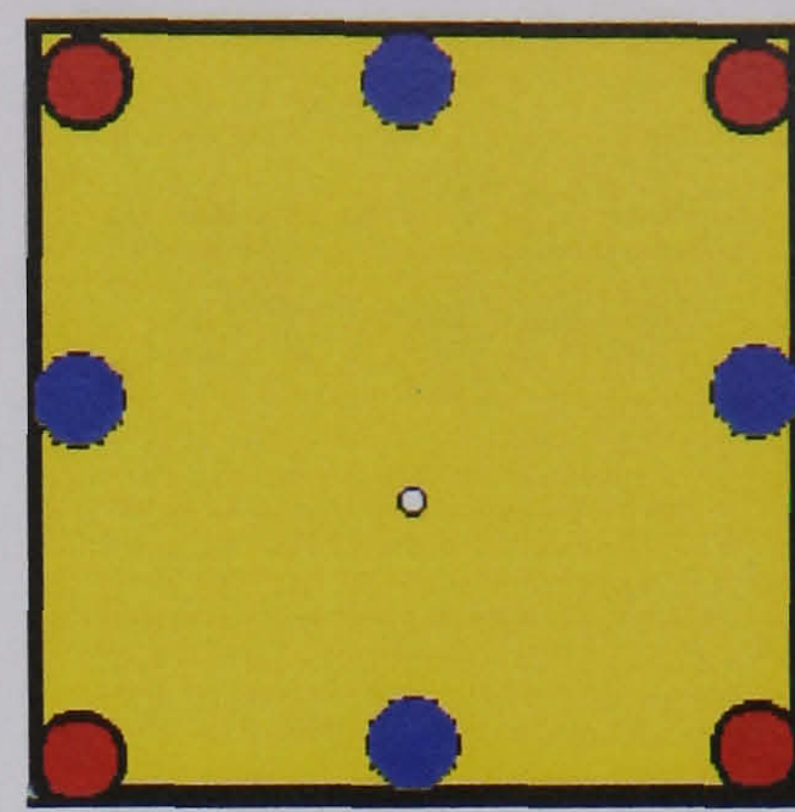


Figure 7.14: Cathy's cross construction

She starts the game.

Cathy: …Eh…Let me copy one to have a look…Ok! Another little ball. Move this a little bit. 1,2,3…Let's start it.

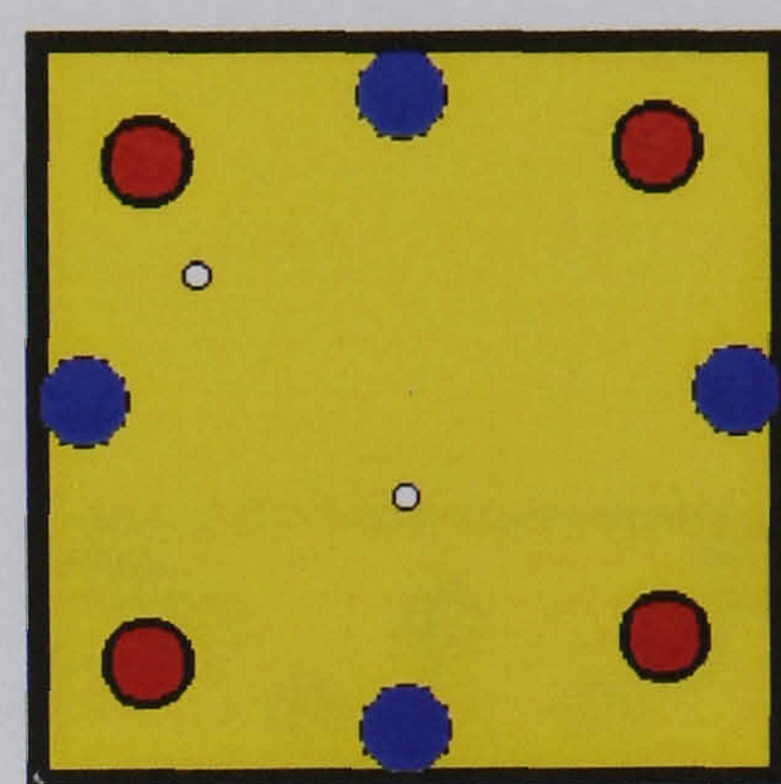


Figure 7.15: Cathy's cross construction developed into a circle

Cathy here first decided to make a pattern, a 'cross' as she called it, and tried it out to see whether it was working. As this did not work in the short-term, she decided to bring the red balls closer, add another white ball, and this developed into a patterned circle. The idea of the circle was for the white balls to touch the red or the blue balls the same number of times. The circular arrangement of balls emerged in an attempt to place the balls equidistantly from the centre. In Anne's (6 6/12 year-old girl) case, a circle turned out to be the starting point for a symmetrical development. Anne started by having the white ball in a circle and then constructed another symmetrical random representation by copying more balls.

Anne: …I'm going to make all the balls have the same size. I'll do another arrangement.

Researcher:	So, what are you doing now?

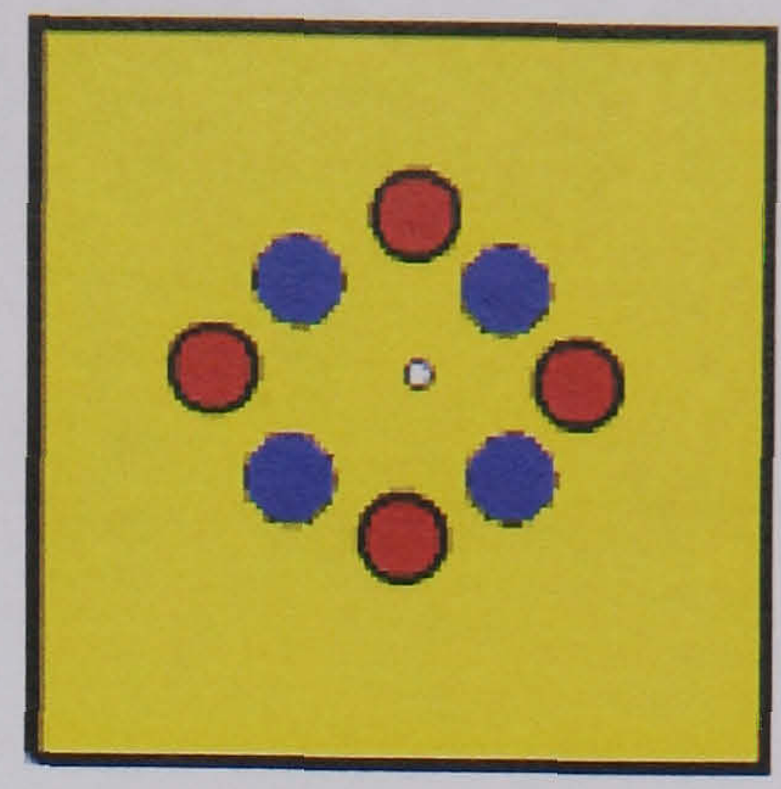


Figure 7.16: Anne's fair circle construction

Anne:	I'll make more copies of them.
	She starts the game.
Researcher:	Oh…does it work?
Anne:	Yeah…It keeps going up, down, up, down….
Researcher:	Ok!
Anne:	I'll make more copies…
	She stops the game.
Researcher:	What's the arrangement now?
Anne:	That one (the blue ball) is facing that one (the red ball) and that one is facing that one and so on… I've got also a better idea! They (the red balls) will be opposite a blue one. There!

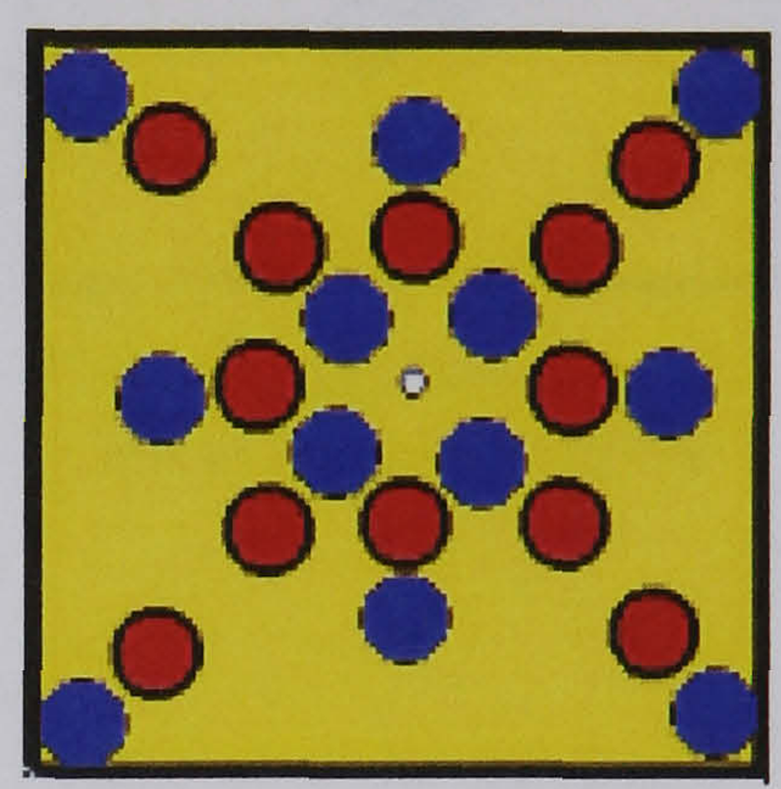


Figure 7.17: Anne's circle construction developed into a patterned symmetrical one

Researcher:	What did you do?
Anne:	The blue ones are facing the red ones and the red ones are facing the blue ones.
Researcher:	Ok! What number will you have here (on the scorers)?
Anne:	I don't know, I'll try it out!
	She starts the game.

Anne here thought about making a circle and placing the balls in a patterned way. Although, her construction was working from the beginning, she decided to develop her sample space into a symmetrical form with red and blue balls facing each other, so that each ball to be selected by the white ball the same times as the others.

Irene (7 6/12 year-old girl) also developed an idea for a patterned circle, when she decided to add more balls to her lottery machine.

Irene: I will get this blue ball, put it where the red was and I will place the red near to it, such as to have blue, red, blue…

Researcher: Why are you doing this?

Irene: I am making it like a pattern because with this way they could have equal points! Is it right?

R: There is no right and wrong… I just need your ideas!

I: Ok… I think now it's ok!

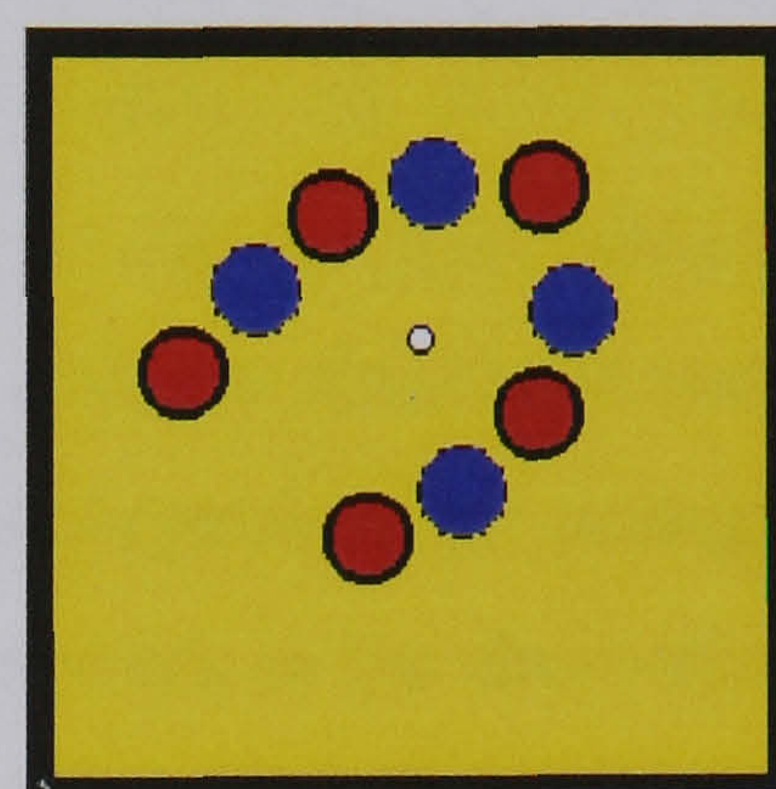


Figure 7.18: Irene's semi-circle fair construction

She starts the game.

I: Oh… you bad red balls! Oh…look! 21-21! Something happens! Look we have equal numbers! (She laughs). They got 61 points!…Oh…it is ok! Wow…102! 104! Oh it moved down… 112-122… Oh, the poor space kid! It moves up now! Oups! A… It seems to move up now… Ahhhhhhhhhh 207-207! *She laughs!*

Irene's construction of a fair sample space empowered her thinking about symmetry and she continued to construct sample spaces by surrounding the white ball in a patterned way.

Irene: I will make something else. I found the way!

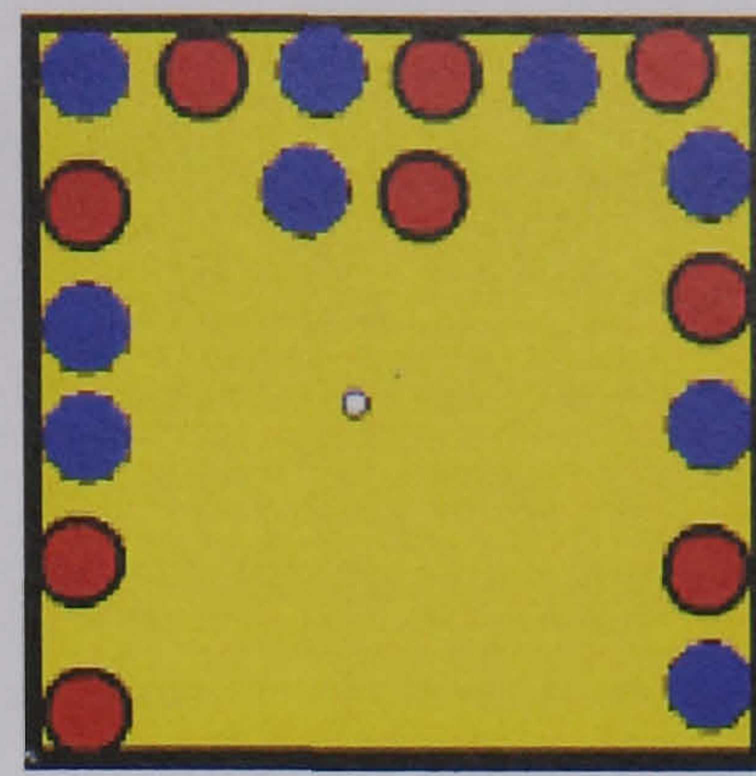


Figure 7.19: Irene's patterned fair construction

Her idea of 'surrounding' the white ball and making a pattern was to 'control' the white ball in a way, touching both blue and red balls the same number of times. As she said 'it is better for both colours to get the same points on the counters'.

It appears from the data that symmetrical placement is a very useful strategy for children to construct a fair environment. Twenty-two out of twenty-three children used one or more symmetrical strategies to construct their fair environment and twenty of them were satisfied with their constructed result. The fact of using so many symmetrical representations for constructing a fair environment shows that the children associated very strongly fairness with symmetry. Where children tried more than one symmetrical strategy in their game, it might be because even if their construction did not work in the short-term they tried to make another symmetrical representation for the construction of fairness. Table 7.1 shows which strategy each child used and how many children of the 23 used each of this strategy.

			Symmetrical Strategy			
Name[24]	**Age (Y: M)**	**Sex**	**Symmetrical balls**	**Symmetrical groups**	**Making Patterns**	**Making circles**
Anne	6:10	F	√		√	√
Anthony	5:10	M	√	√		
Brian	6:6	M		√	√	√
Cathy	7:6	F	√	√	√	√
Chris	7:8	M	√	√	√	
Demis	7:5	M	√	√		
Fiona	7	F	√	√	√	
George	6:8	M				
Helen	7:6	F	√	√	√	
Irene	7:6	F	√	√	√	√
Jane	6:7	F	√	√	√	
John	6:10	M	√	√		
Karen	7:3	F	√	√		√
Lucy	7:8	F		√	√	
Mathew	7	M	√	√	√	
Nichol	7:8	F	√	√		
Orestis	7	M		√		√
Paul	6:10	M	√	√	√	√
Rachel	7:3	F	√	√	√	
Simon	7:10	M	√	√		
Tom	7	M	√	√		√
Victoria	6:6	F	√	√		
Zeta	6:4	F		√		
The total number of children (out of 23)			18	20	12	8

Table 7.1: The total number of children constructing different symmetrical strategies for fairness

Table 7.1 shows that the children tended to construct more than one symmetrical strategy in their game, or used a combination of strategies. Symmetrical balls and symmetrical groups were the two strategies that most of the children used for constructing a fair environment, and as the data show 22 out of 23 children used at least one of these two strategies. The children who constructed a symmetrical balls strategy tended also to construct a symmetrical groups strategy, after they had linked local and global events. It was also noteworthy that of the eight children who constructed a 'making circles' symmetrical strategy, five of them also constructed a strategy of making patterns. The data presented here also show that children instead of looking for symmetrical ways to describe the random behaviour developed symmetrical representations for constructing a random fair environment. Table 7.1 also shows that one child, George, did not express any symmetrical strategy, but he constructed an asymmetrical one for expressing fairness. The

24 All names of the children are pseudonyms.

following section illustrates another way of representing fairness that was developed by using asymmetrical representations. The table also shows that the sex and the age of the children did not influence their decisions of constructing unfairness in their game.

7.3.2 Asymmetrical spatial representations of fairness

Table 7.1 shows that 22/23 children tried to construct a symmetrical spatial representation for the construction of a fair environment. It also shows that 22/23 of them used multiple strategies. Fifteen out of twenty-three children also used asymmetrical spatial representations of fairness, basing their decisions on short term and long term movements of the ball. These strategies had the following characteristics: a. equal number and size of two balls (7/23) and b. mixed up balls (11/23), which can be also divided into: equal number and size of balls (10/23) and different number and/or size of balls (7/23).

7.3.2.1 Representation of fairness with an equal size of two balls

This strategy occurred when children wanted to have two balls in their sample space and they were not concerned about their spatial arrangement. The fair environment played a major role in helping Jane (6 7/12 year-old girl) to recognise the importance of whether two balls were equal in size or not. As she said 'if one of the balls is bigger, it (the white ball) will touch the most of the time, because the ball takes up more space in the yellow square'. Her criterion for whether the two balls had the same size is if they get the same score, and thus the game was fair.

Jane:	I think the red will win.
Researcher:	Why is that?
Jane:	I think I made it a little bigger than the other… We can start the game and if the scorers are the same that means they have the same size, otherwise the one is bigger than the other.
R:	What about the space kid?
J:	If it is as now (on the yellow line) that means our balls have the same size…

She starts the game.

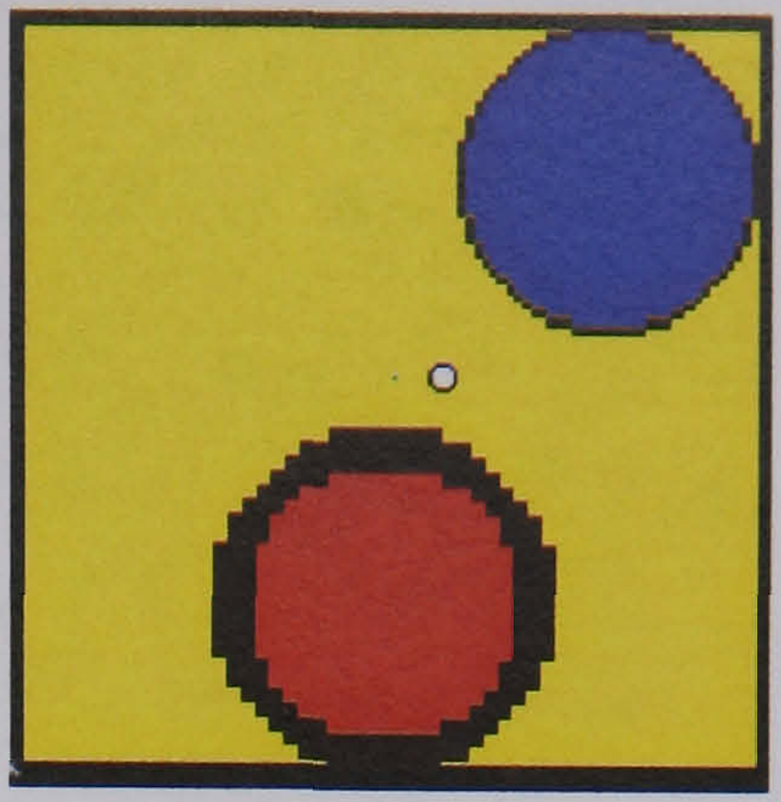


Figure 7.20: Jane's asymmetrical fair construction

J: You see, now it means that they have the same size. Our space kid is near the yellow line.

Jane looked at the effects of the outcome of the lottery machine and she used the global events, the scorers and the space kid, to see whether her environment was fair. She made a connection between the spatial appearance of the sample space and the possible outcome from the game in the longer term. Her construction implies that she realised the arbitrary movement of the ball and she was not concerned with where to place the two balls, but to make them have the same size.

The children who used this strategy based their idea on the view that there are two events in the game, so by having two equal sized red and blue balls in our lottery machine we achieve fairness (the two events become equiprobable). Fairness was a 'data condition' in their game and was used by them to be sure that the two balls were equal in size. In this construction fairness was achieved by the placing of the balls, controlling the size of the balls and making connections between the spatial appearance of sample space and the possible outcome of the game in the longer term. The eight children who used this strategy seemed to find it easier to handle two global events by having only two local events in their lottery machine, only two balls, and they decided to make these balls bigger than the default size, but the number to stay the same as the global events of the game. Next, I will describe constructions, which mix up more than two balls to produce a fair representation.

7.3.2.2 Representation of fairness with mixed balls

Mixed balls constructions have more than two balls in the lottery machine and show a different kind of thinking about the two global events. Children in this case did not connect the number of the global events with the number of the events in their lottery machine and they used more than two balls in their lottery machine and mixed them up. The children developed this strategy after they watched the continuous arbitrary movement of the white ball and realised that the white ball does not follow a patterned movement. Two sub-categories of this strategy can be identified, based on a. equal number and size of balls (used by 7/23 children), or b. different number and/or size of balls (used by 10/23 children).

7.3.2.2.1 Equal number and size of balls

The case of Paul (6 10/12 year-old boy) is an example of a construction of equal size and number of balls of each colour. He used more than one ball of each colour, but he tried to make each colour to have the same number and size between the balls.

Paul: We have to make all of them bigger. All the balls to be as big as that one. I will make this bigger. They (the scores) may get the same points…it (the space kid) may go down. I will switch on the game now.

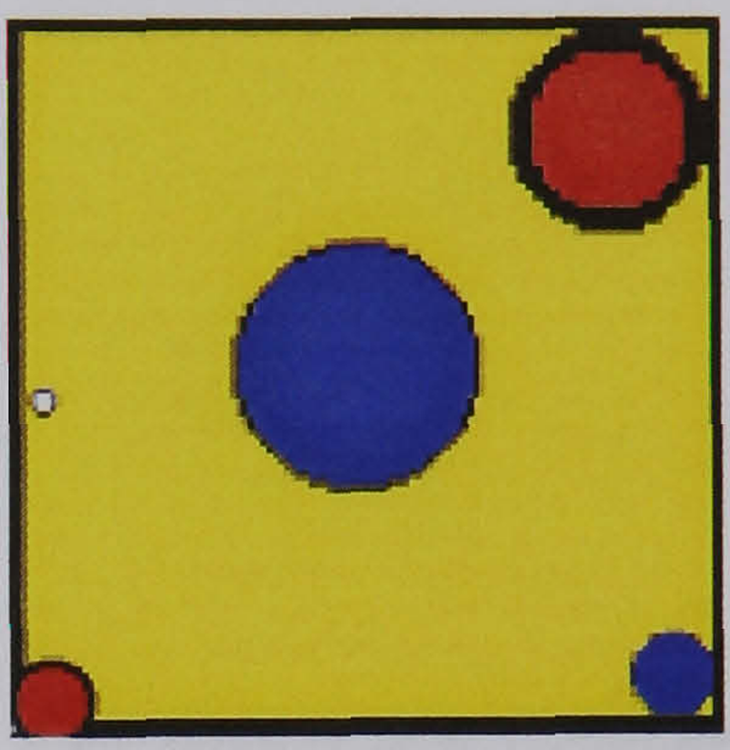


Figure 7.21: Paul's mixed asymmetrical fair construction

Since the number of the balls was more than two in this case, the arrangement in the sample space played a role, as the white ball could be blocked in amongst and thus generate unequal points. Perhaps this is the reason why Paul was unsure whether this construction would bring a fair result.

In Helen's (7 6/12 year-old girl) case there is a mixture of large and small balls.

Helen: … Move some balls… Ok! We have two big red balls and two big blue balls, three little red balls and three little blue balls. Now, I think they will be equal.

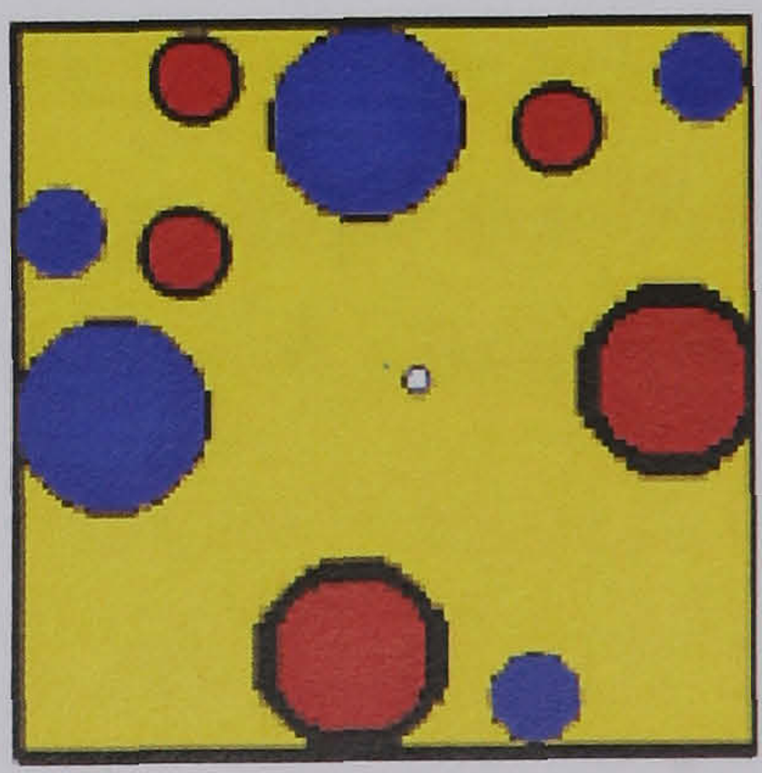


Figure 7.22: Helen's mixed asymmetrical fair construction

Researcher: How did you arrange them?

H: Let's say it move to the red first and then to the blue, it will move like this, it will get the red again and then the blue. When it goes to the red it will go to the blue as well. It will get both colours.

Helen constructed here two big balls of each colour and three small balls of each colour. She placed them in a way that the movement of the white ball could reach each ball the same times and she suggested a possible movement of the white ball. She mixed them up by keeping the number and the size of the balls balanced for the two teams and she considered how the place of each ball would influence the movement of the bouncing ball.

Lucy also based her decision on the movement of the white ball.

Lucy: It (the white ball) will go all the way round. Let's say the white ball will move right, left, in the middle, up, down, on the edges and it will touch all the balls. We have ten balls now... The scorer will get many points.

Researcher: Which scorer?

L: The red or the blue one.... They might get the same scores. They will get many points. The balls are mixed up.

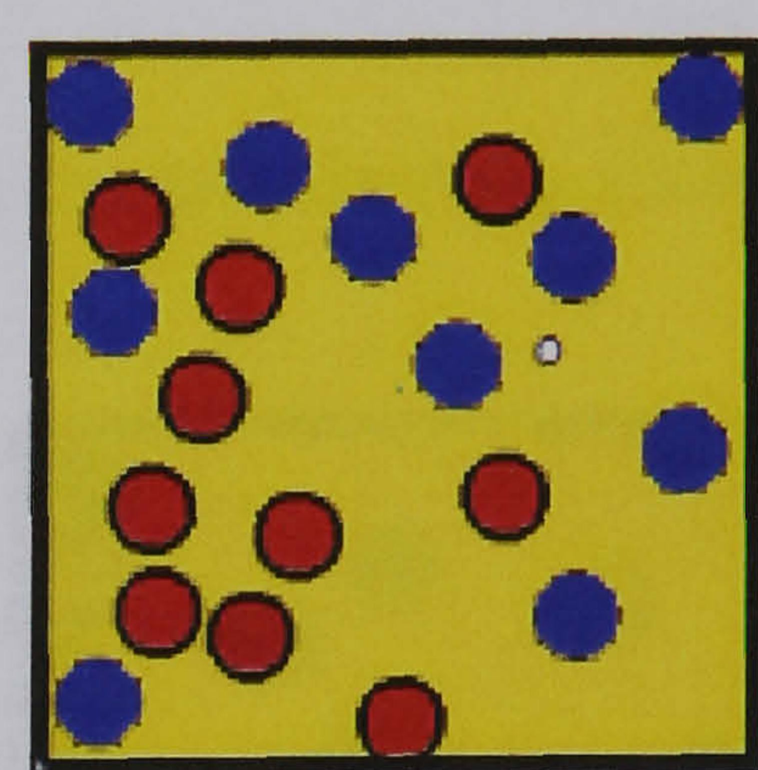


Figure 7.23: Lucy's first mixed asymmetrical fair construction

Since Lucy could not find a particular pattern of movement of the white ball, she took the same number and size of blue and white balls and mixed them around. This construction did not get a fair result, because the white ball got blocked between balls and it could not reach all the balls evenly. This led Lucy to make a further construction of mixed balls, which included many duplicate white balls.

L: I mixed them up…I put them just in a way to be equal. …I will copy some more white balls, here and here. Now, they might get all the balls the same times.

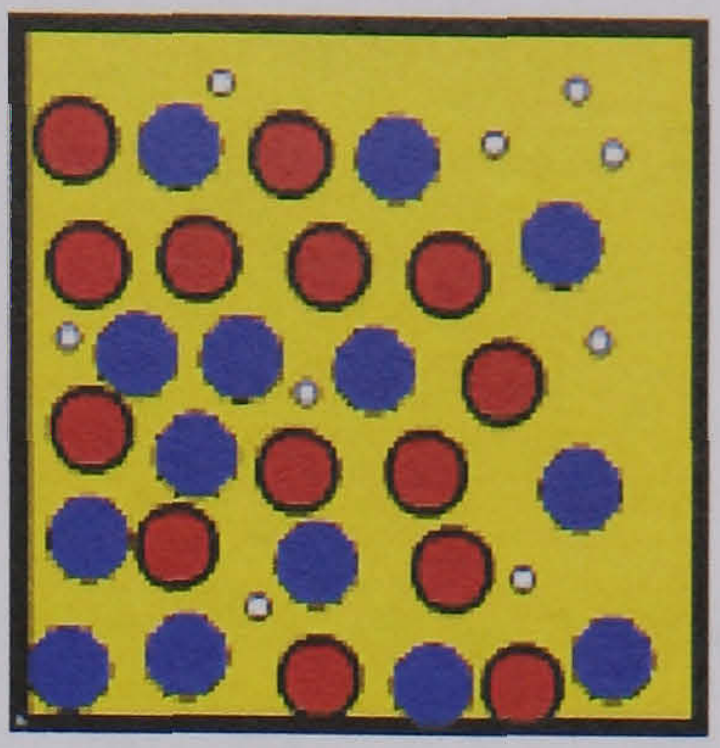


Figure 7.24: Lucy's second mixed asymmetrical fair construction

Lucy in this construction tried to 'block' the white balls in different places in order to get all the 'unreachable' balls. The construction worked over the long term, to Lucy's delight and satisfaction. It seems that the movement of the bouncing ball influenced children's idea of mixing balls. The idea of 'unreachable' balls that was expressed by Lucy is something that Simon (7 10/12 year-old boy) described as having 'a sabotage' in his lottery machine. This effect made Simon and other children try to construct a fair environment with different number or/and size of balls, as is described next.

7.3.2.2.2 Different number and/or size of balls

Simon made a fair environment by placing the balls around the space and thinking about which balls were making 'sabotage' to the others.

Simon: I think we are going to get equal numbers. Let's see…

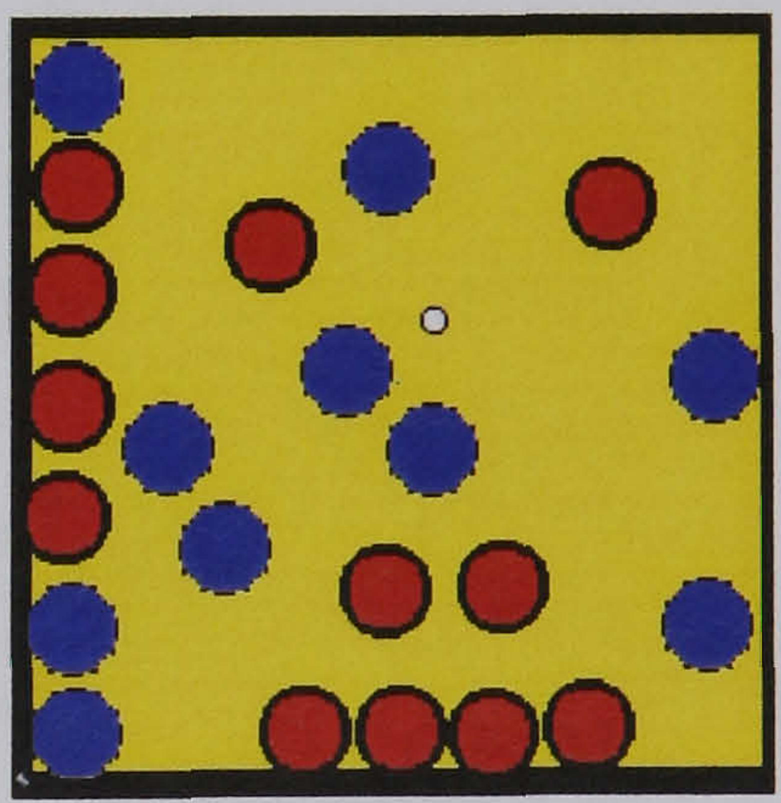


Figure 7.25: Simon's mixed fair construction

He starts the game.

S: Oh….look they got equal numbers…

Researcher: I don't understand why. The reds are more than the blues.

S: It doesn't matter how many they are. What it does matter is the shape. One ball might sabotage another one.

R: 'Sabotage'?

S: You see here this blue ball...when it (the white ball) goes like this it will touch that one and not get the red one...

Because of the spatial representation of sample space Simon developed a strategy where the number and size of the balls was not the characteristic of a fair environment, but the place of the balls also played a major role. So, he constructed a sample space with 12 reds and 10 blues, with all the balls having the same size, and placed them in a way that made him expect that the two colours would get equal points. Simon's construction shows evidence of thinking about how each event in the sample space could have a different probability to occur. As he described, the repetitive position of a ball changed the probability of a ball being hit. By 'sabotaging' some balls in his sample space, Simon decreases the probability of these events to occur and creates events in his lottery machine with different probability to occur. This is an example that shows how children 'reinvented' the probabilistic idea of *distribution.* Simon's idea of changing the probability of the events in the lottery machine was used in order to make the game fair by having unequal numbers of coloured balls.

George (6 8/12 year-old boy) made a more 'strict' mixture of balls that was also influenced by the 'sabotage characteristic'. He constructed inside his sample space a pyramid, without worrying about the number of the balls of each colour.

George: ...I have a very good idea. I will make... you will see in a while. I will make a shape... You will see. I will put this here... I am making a shape.

Researcher: Which shape?

G: Something that I forgot its name.

R: Ah! Is it like a pyramid?

G: Yes! I will do it like this. The blue will get the first point and first our space kid will move down and when it will touch the red it will move upwards.

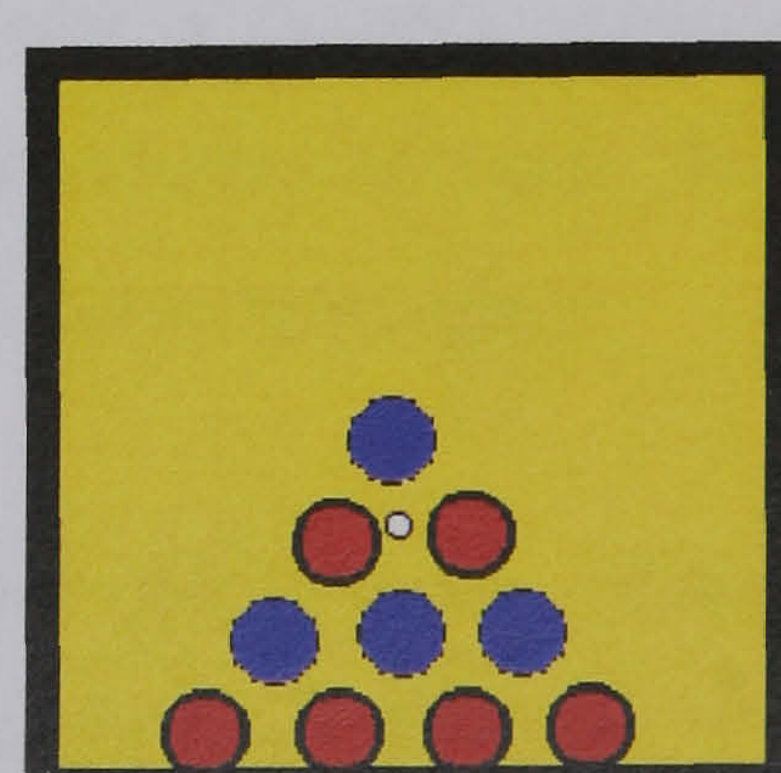


Figure 7.26: George's mixed pyramid

Although George placed more red balls in his construction, he placed them in a way that some coloured balls would prevent the white ball from touching other balls. Thus, he expected to achieve fairness in his game. George's case is another construction having an unequal number of balls, which have different probability to be hit (the idea of distribution), and as a global outcome to have a fair result in the game.

Another construction of an asymmetrical fair environment was Tom's (7 year-old boy). He did not have the same number of balls of each colour, and attempted to compensate the 'spatial' imbalance by modifying the size of the balls.

Tom: I know. To put all these up…

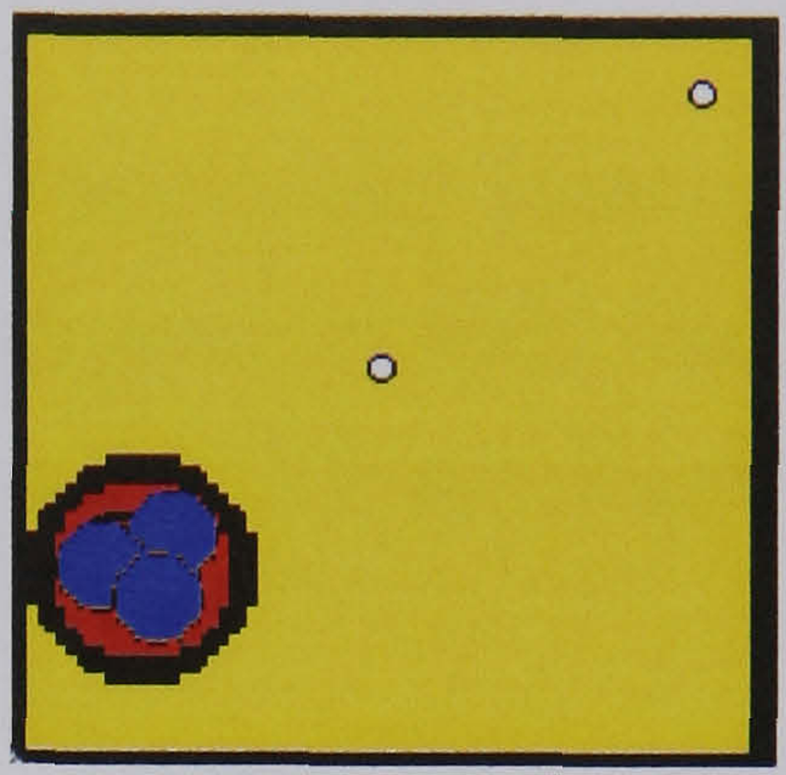


Figure 7.27: Tom's asymmetrical fair construction

R: How are you arranging them?

T: When it goes to the red to get the blue as well.

Tom tried to place one colour on the top of the other, as he thought that was the easiest way for the white ball to get both colours at the same time. It is worth mentioning that the starting point for Tom's construction was for all the balls to have the same size in the lottery machine (as is described in Chapter Five, iteration 3). His idea showed that he did not think of equality of the two colours in terms of the number of balls, but in terms of equalising the space that each colour occupied inside the sample space. So, Tom tried to achieve equality in space by having the two colours occupying the same space. He made three blue balls to be equal with one red ball. Tom seems to make with his construction a number-space connection. Each blue ball in the game had a different probability to occur from that of the red ball, and this led Tom to have a different number of balls, thus achieving an equiprobable global outcome of the two events, and fairness in his game.

The representation of fairness with mixed balls was expressed after children had realised the unpredictable continuous movement of the white ball. The ten children expressed this construction either by having equal number and size of balls or different number and/or size of balls. The goal of this strategy was for the two events to have equal probability distributions, achieving an equiprobable global outcome of the two events. The following section describes the strategies that children used to construct unfairness. When children tried to construct fairness they also faced the idea of unfairness when their fair construction did not work. They also had to think of unfairness explicitly when they faced the problematic situations of one team to get more points than the other, or the space kid to reach one of the two planets.

7.4 Unfairness

The construction of unfairness can be divided into the following representation forms: a. different number of balls, b. different size of balls and c. spatial arrangement of balls. Also, the children had to construct an unfair environment when they engaged in the problematic situation of certain and impossible events.

7.4.1 Unfairness due to different numbers of balls

The children seemed to understand that having more balls of one colour makes the game unfair. John (6 10/12 year-old boy) made a construction of an unfair environment for the blues, his team, to win.

John: Ok! I will put this here. Ok!

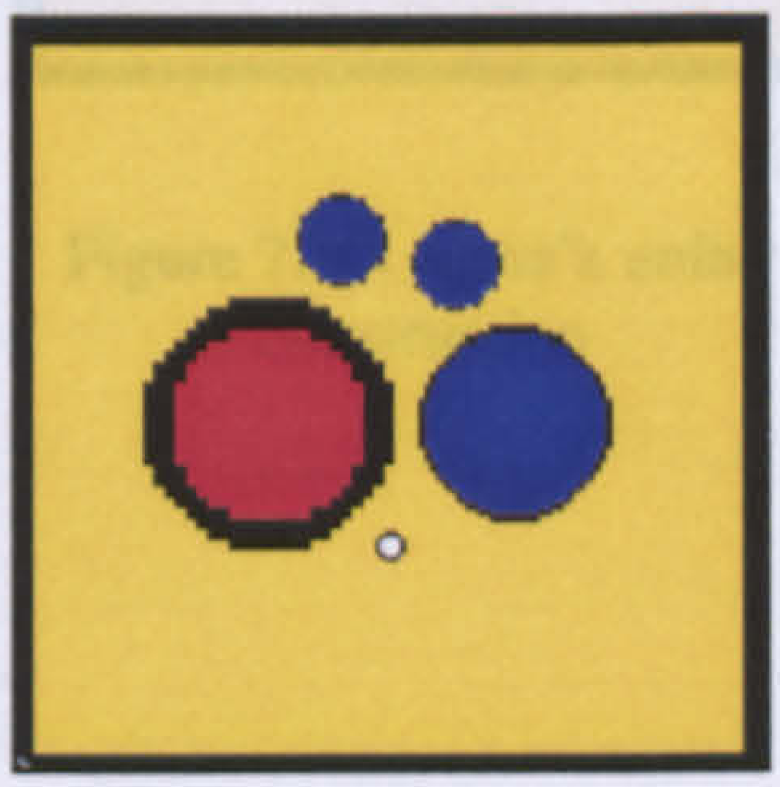


Figure 7.28: John's unfair construction

Researcher: Which colour is going to win?

J: We won't get equal points! Well, you know, we are with the blues because there are more blue balls inside the sample space now.

John constructed his sample space for the blues to win, with one large ball for each colour and another two small blue balls. It seemed to be easy for John to connect unfairness by having an unequal number of balls.

7.4.2 Different size of balls

The space that each ball occupied in the sample space, the size of each ball, was a criterion for children to judge an unfair environment. Brian explained in the following arrangement:

Brian: The ball will get more points because it's bigger. It will get more points. It has much bigger place and this is very small and if it goes like this it will touch it and like this it won't touch it. It can get more points faster than the red one, because it's bigger. If it goes like this and like this it will get more points.

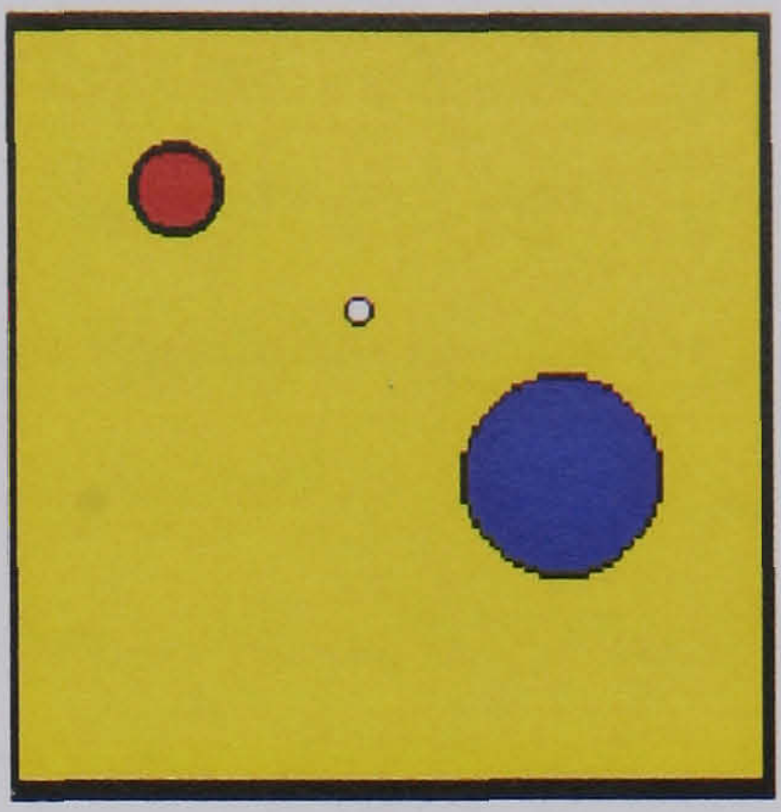


Figure 7.29: Brian's unfair construction

So, in an unfair environment the winning ball is bigger than the other- it takes more space and it gets points faster than the other ball. Anne also described the importance of the occupied room and space inside the sample space in the following snapshot.

Anne: The blue balls are bigger.

She laughs.

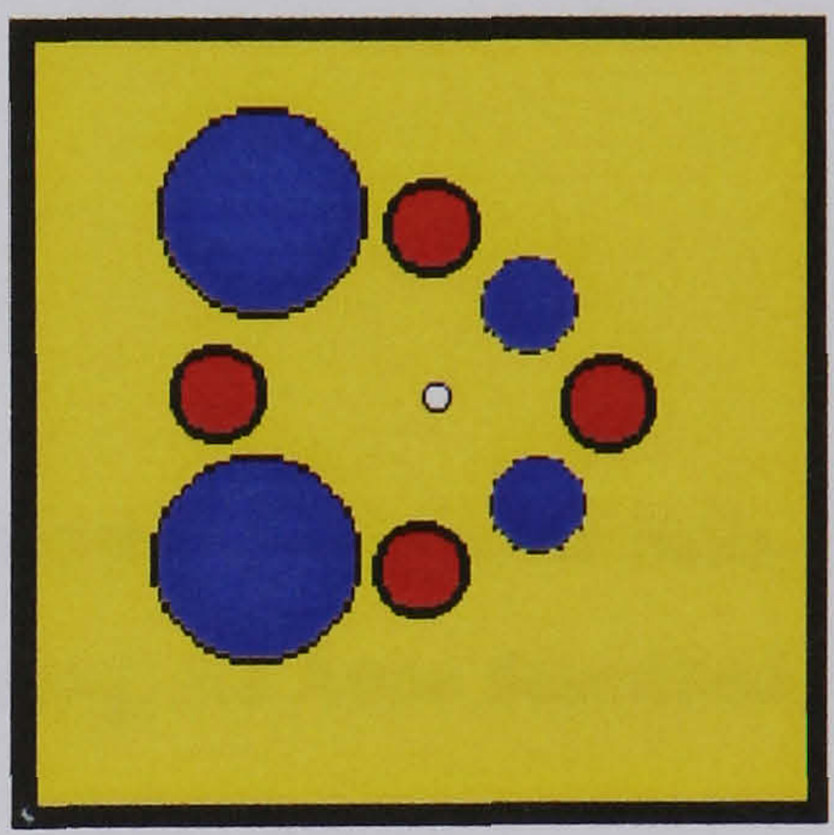


Figure 7.30: Anne's unfair construction

Researcher: Is it going to be fair now?

A: I don't think so…It's not!

R: Why? I have four blue balls and four red.

A: Because these two blue balls are bigger and they take much more space. The ball can hit them more because they have more space.

R: What do you mean by 'more space'?

A: It means is more full. So, the ball will mostly touch those ones.

R: What does it mean if you have more space in the lottery machine?

A: It means like cheating…

In Anne's construction there was an equal number of blue and red balls, but the two blue ones took more space around the circle. Anne here made, like Tom above, a number-space connection. But, in Anne's situation the number stays the same and the space is used for increasing the probability of an event. This made it obvious for Anne to describe this sample space as unfair and that the blue balls are going to win. As she said, when the balls take more room in the sample space the result is like cheating. She also described the unfair environment as a 'cheating' one when she made a spatial arrangement of the same size and number of balls. Anne worked on her fair representation of having a pattern in a circle and then unbalanced the fairness by having the same structure, but increasing the space of two balls. She seemed to realise that when you unbalance the distribution, you unbalance the outcome as well. It seems that her idea of distribution made her also think of the possibility of keeping the size and number of the balls the same and to change the probability of the events by changing the place of the balls. The following section describes how Anne changed her construction by changing the spatial arrangement of the balls to achieve unfairness.

7.4.3 Spatial arrangement for unfairness

Spatial arrangement was a strategy where children had the same number and size of balls, but they arranged them in a way that they made them feel like cheating. As Anne described

Anne: I want to make it easier for the space kid to go up...

Researcher: What are you doing?

...

R: I don't understand...I need you to explain to me.

A: I have four balls...so, I'm blocking them.

R: What about the red balls?

A: I'm going to do that and the white ball is going to hit the balls and go up, up, up...(the space kid will move upwards).

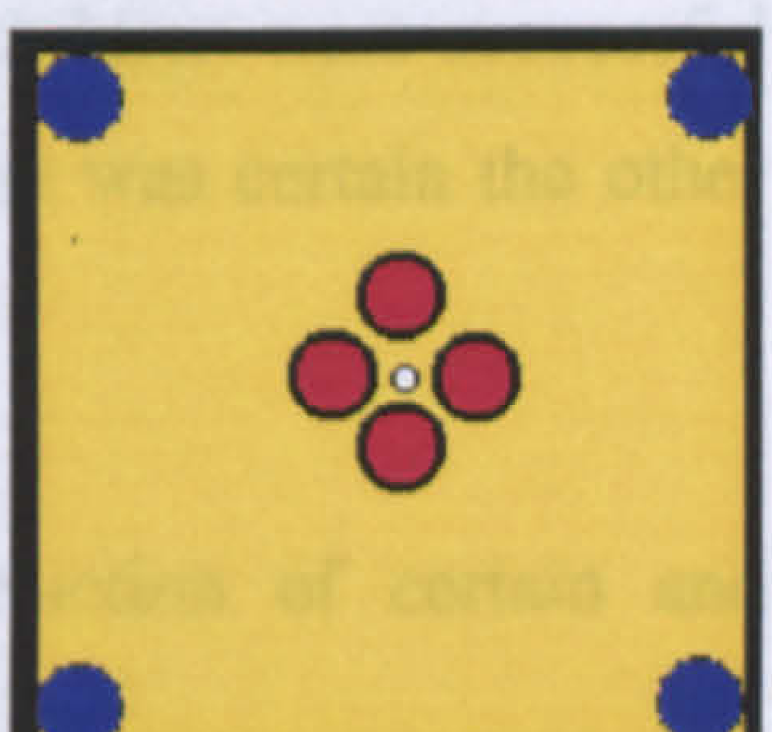


Figure 7.31: Anne's spatial arrangement for unfairness

R: Why did you put the same number of blue balls?

A: So, not to be a cheating game. If you don't have the same numbers it's like cheating!

R: So, you don't cheat...

A: It's *like* cheating, but not exactly!

Anne constructed an unfair environment here, by blocking the white ball inside a red circle. As she said, this arrangement would make the space kid move up easily. Although she placed the same number and size of balls, she believed that the way she arranged the balls inside the sample space was an unfair representation. It is again 'like cheating', but not exactly, as she said, because the number and the size of the balls are equal, this might be the reason of not deleting the blue balls. She seemed again to realise that when she unbalanced the distribution, she unbalanced the outcome as well.

The following subsection describes another case of unfairness, for certain and impossible events, and illustrates the children's representational forms for constructing these events. For constructing unfairness, the children did not only employed strategies of changing the number of balls, but also of changing the probability of each event to occur by keeping the same number of balls.

7.4.4 Certain and Impossible events

The children's constructions of certain and impossible events were employed in the representational forms for which there were not events of the one or the other colour, depending on for which team an outcome would be certain or impossible. Certain and impossible events were necessary when they faced problematic situations like 'what could you do in order the space kid get to the one planet very quickly' or 'what could you do if you wanted the blues not to get any points'. All the twenty-three children were successful on certain and impossible events and they realised that if one event was certain the other event is impossible to be occurred.

Karen (7 3/12 year-old girl) expressed a view that the construction of certain and impossible events for her was very easy. As she said

Researcher: Can you do something to get to the red planet quickly?

Karen: That's easy. I will use the ball. I will leave 1 red ball and 10 blues. Is it on the blue that we want it (the space kid) to go?

R: No, it should move quickly to the red planet.

K: Then, let's have only one blue, just not to be too mean, and the others to be red. I will leave three white balls to move quickly!

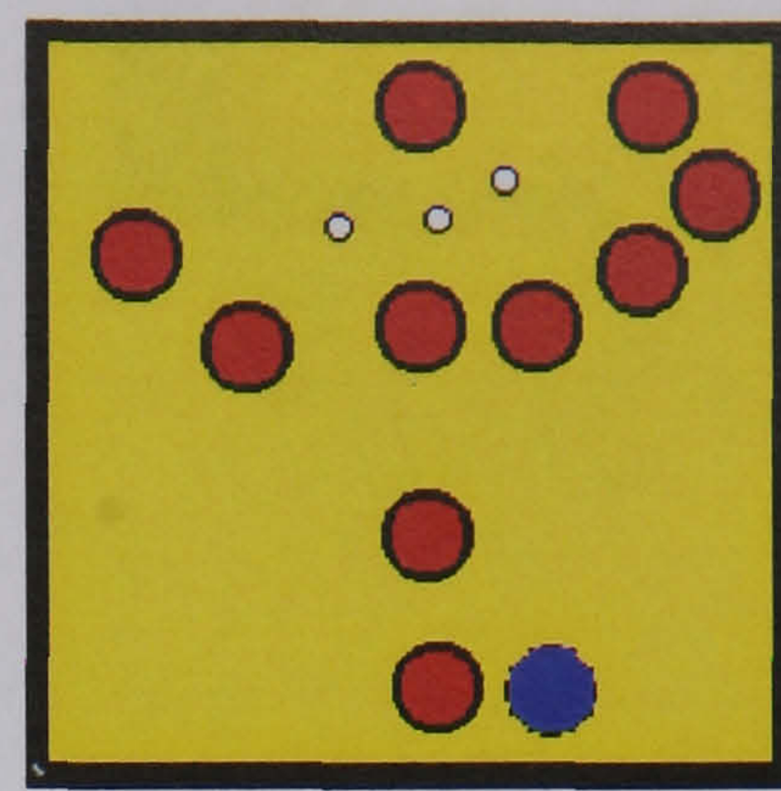


Figure 7.32: Karen's construction with many reds and one blue ball

She starts the game.

K: Look! Wow… it got to the red!

The game stops.

R: If we want the blues not to get any points?

K: Then we don't want any blue balls.

She takes away the blue ball.

R: Are you sure about it?

K: Very sure… You want to have a look? Look!

She starts the game.

Karen's idea was first to have only one blue, just 'not to be too mean' as she explained, but she placed this ball in a position where it was not easy to get a point. However, as blues got some points in the game, she decided not to have any blue balls and then she was very sure that it was impossible for the blues to get any points.

Demis (7 5/12 year-old boy) started his constructions the other way around. He wanted for the white balls to be impossible to touch any red balls and to be certain for the space kid to move down to the blue planet. As he said

Demis: … Let's do something else. I will destroy some objects… I will not use any bricks. I will move all the reds…

Researcher: Why?

D: In order not to touch the red balls…the white ball to touch the blues and then (the space kid) to move down. This is easy…

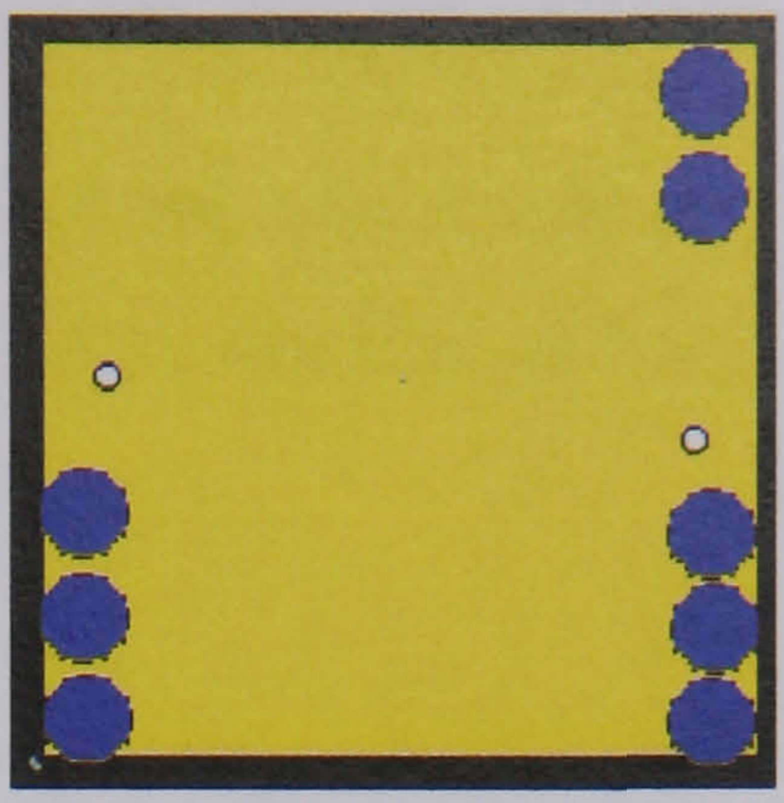


Figure 7.33: Demis' certain/impossible construction

He starts the game.

R: Will the space kid move upwards as well?

D: No, there are not any red balls. That's it!

R: Ok… Now, let's say that we are not allowed to have no reds, but again we need our space kid to move on the blue planet.

D: Only to move down?

R: Yes, what will you do?

D: Ok…

He stops the game.

D: I will put a red ball on the site. That's it.

He starts the game.

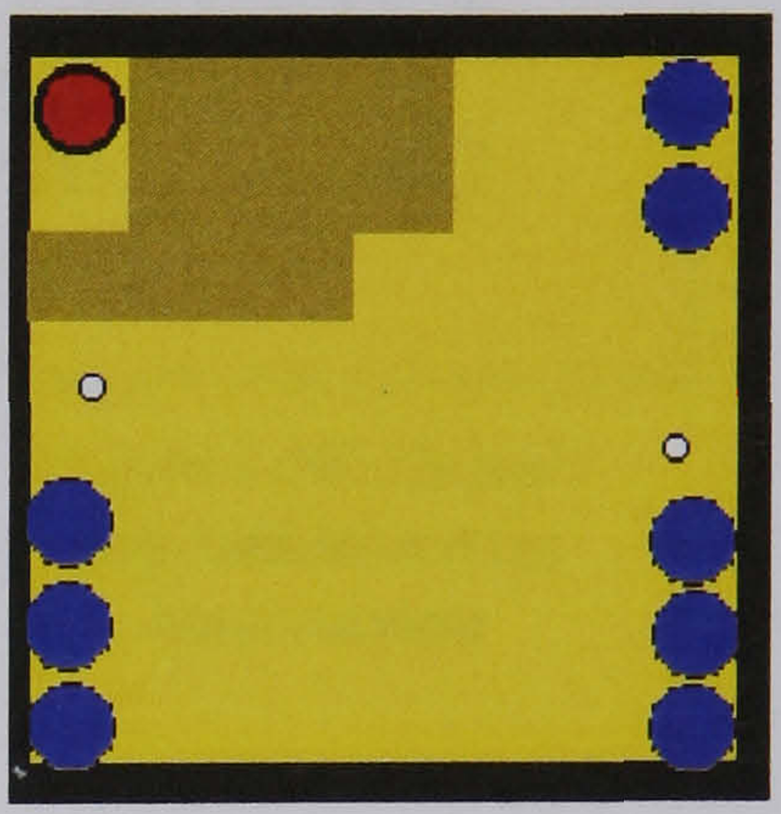


Figure 7.34: Demis' construction with one red and many blues

R: We got some red points…

D: But, it moved down! …

D: We cannot have reds inside and not to get any reds! How this could happen? I couldn't do it! I will do something else… I don't need any bricks, but I will put blue balls here. That's it!

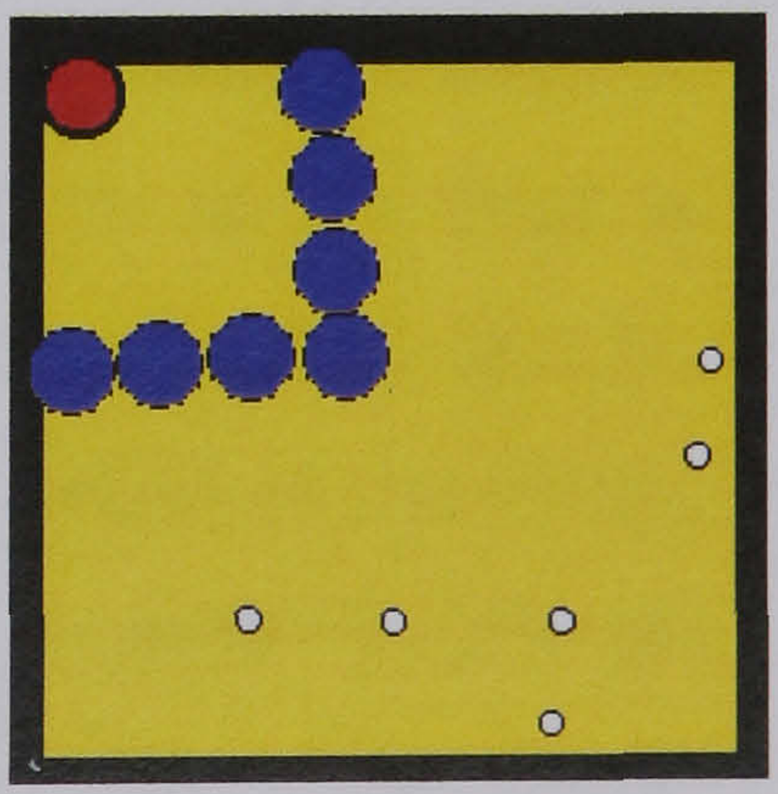


Figure 7.35: Demis' second construction with one red and many blues

For Demis it was obvious that with no red balls inside the lottery machine, it was impossible to get any red points and thus, it was certain that the space kid would move down. With a red ball inside his lottery machine, even having decreased the probability of getting red in his distribution, he also expressed the idea that he could not be sure that the reds would not get any points, as he explained that he could not have reds inside the lottery machine and not get any reds.

Helen (7 6/12 year-old girl) was also clear that in order not to get any blue points there should be no blue balls.

Researcher:	Ok… Can you do something in order that our space kid move straight away to the red planet?
Helen:	Yes! I will take away all the blue balls and then it will not touch any blue balls, so it will move upwards.
R:	Are you sure?
H:	Yes! It will not move down. There are no blue balls. No blue balls, no points!

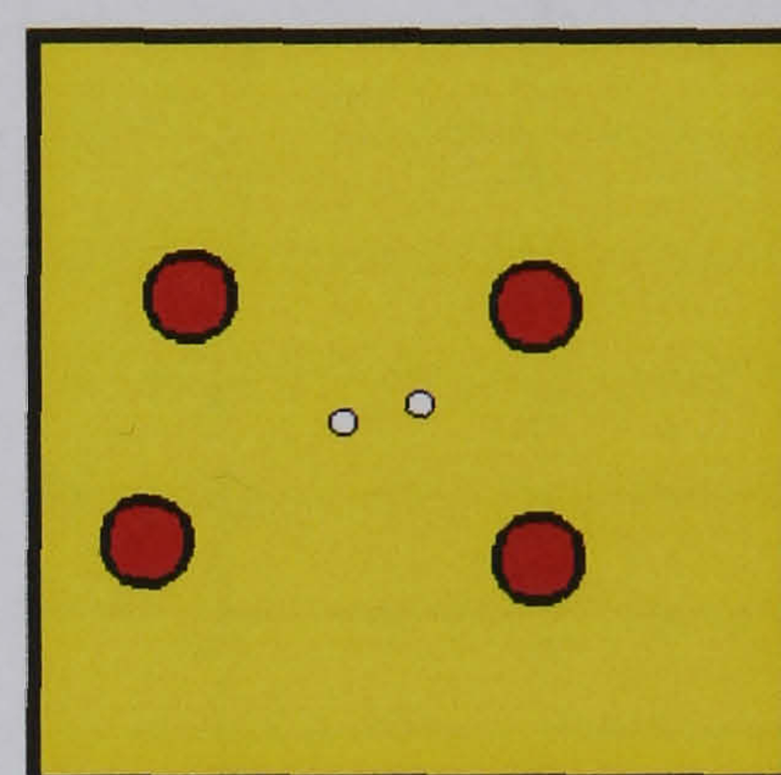


Figure 7.36: Helen's impossible/certain construction

R:	Do you want to try it out?
H:	Yes, I am sure! (*She laughs*)

As Helen described, no blue balls means no points and this representation made it sure for her that the space kid would move straight away to the red planet.

As the above episodes indicate, all the children constructed very easily representations for impossible and certain events. They also grasped that if the probability of one event is certain then for the other it is impossible. They also expressed a distributional idea in, where they can have both colours in their game, but decrease the probability of one colour to be selected. But, as they explained, decreasing the probability of an event is different from the impossibility of an event to occur. Table 7.2 shows a global view of the use of each strategy from each child.

Name[25]	Age (Y: M)	Sex	Unfairness: Different number of balls	Different size of balls	Spatial arrangement	Certain and Impossible events
Anne	6:10	F		√	√	√
Anthony	5:10	M	√			√
Brian	6:6	M	√	√		√
Cathy	7:6	F	√	√		√
Chris	7:8	M	√			√
Demis	7:5	M	√		√	√
Fiona	7	F	√	√	√	√
George	6:8	M			√	√
Helen	7:6	F	√	√	√	√
Irene	7:6	F	√	√	√	√
Jane	6:7	F	√	√	√	√
John	6:10	M	√	√		√
Karen	7:3	F	√		√	√
Lucy	7:8	F	√	√	√	√
Mathew	7	M	√	√		√
Nichol	7:8	F	√		√	√
Orestis	7	M	√			√
Paul	6:10	M	√	√		√
Rachel	7:3	F	√	√		√
Simon	7:10	M	√	√		√
Tom	7	M	√			√
Victoria	6:6	F	√			√
Zeta	6:4	F	√			√
Number of children (out of 23)			21	13	10	23

Table 7.2: The total number of children constructing different strategies for unfairness

Table 7.2 summarises the strategies that each child developed for the construction of an unfair environment. It shows that all the twenty-three children tended to construct an unfair construction for certain and impossible events. It also shows that 21/23 children used the strategy of having different number balls in constructing unfairness and 13/23 children used the spatial construction either by changing the size of the balls, or by having a spatial arrangement of the balls. Table 7.2 also shows that children who constructed different size of balls also constructed different number of balls, for achieving unfairness in their game. It can be said that there was a 'link' here between the number of the balls and size of the balls in the spatial lottery machine. The table also shows that the sex and the age of the

[25] All names of the children are pseudonyms.

children did not influence their decisions of constructing unfairness in their game. The next paragraph illustrates some provisional findings of Chapter 7.

7.5 Summary of Chapter Seven and provisional findings

It seems that fairness is a major characteristic in children's games of this age and a desired characteristic, especially when it concerns two different teams. The twenty-three children's representations for constructing fairness and unfairness in the lottery machine fell into two main categories: symmetrical and asymmetrical. Almost all of the children (22/23) expressed fairness with symmetry and twelve out of twenty-three children also expressed fairness with asymmetry. It is significant that the children who shifted from symmetric to asymmetric constructions seemed to be stimulated by their interaction with the game. For example, in Simon's case (as has been illustrated in the previous paragraphs) there is first an expression of constructing fairness in a 'strict' symmetric way, but after interacting with the lottery game he also expressed the idea of having a mixed fair construction where each event in the sample space could have a different probability to occur. In general, after recognising the arbitrary movement of the white ball in the lottery machine, fifteen out of twenty-three children seemed to care less for strict spatial symmetry. It also appears to be a distinction between knowing from before what brings fairness in the game, like symmetry, and developing an asymmetric arrangement by checking whether or not brings fairness in the game. The episodes analysed in this chapter lead to the following provisional findings:

Provisional finding 7.1: The results show all the children's thinking tended to move from finding and describing outcomes to constructing models of fair and unfair random behaviour. For the construction of fairness twenty-two children initially expressed their intuition that symmetry represents a fair situation, with an exception of George who expressed asymmetric fairness. Symmetric fairness was built as a result of what the children already had in mind of what is fair, and that there is a connection between fairness and symmetry. Eighteen of the children constructed fairness by having balls on opposite sides of sample space, equidistant from the centre. Twenty of the children also tried to have an equiprobable sample space by dividing it into two parts one for the one event and one for the other. The animated lottery machine gave them the opportunity to judge their ideas through the lens of the global outcomes. Thus, the manipulations and continuous movement of the game afforded the children a concrete instantiation of their intuitions, and thus an opportunity to 'debug' and develop them. The symmetric fairness seems to be in line to historical evolution where symmetry was connected with ideas of probability. This is analysed more globally in Chapter Nine, section 9.4.3.

Provisional finding 7.2: Where children shifted from symmetric to asymmetric constructions of fairness and unfairness they implicitly used the idea of distribution in order to change the likelihood of an event occurring. In constructing fairness fifteen of the children seemed to 'reinvent' the idea of equiprobable distribution. It appears that the children found a way to achieve fairness by 'unbalancing' their sample space. It can be said that there was expressions about the idea of distribution in ways that are related with asymmetrical constructions. Ten of the children also made a connection between the spatial appearance of the sample space and the possible outcome from the game in the longer term. They constructed asymmetric fairness and they judged their construction based on the global events of their game.

Provisional finding 7.3: In constructing unfairness the children realised that by unbalancing the probability of an event increases or decreases the likelihood of an event to occur. The game encouraged the children to base their construction of unfairness not only on changing the probability of an event by increasing the *quantity* of events, but also by increasing the *likelihoo*d of an event, by changing the distribution. Twelve of the children seemed to connect distribution with the size, the amount and the place of each coloured ball in the lottery machine. The children needed to express distribution in their constructions of the lottery machine and they had a tool to do so.

There were cases in this chapter in which children, while they were expressing ideas of fairness and unfairness, expressed also quantitative ideas of randomness (see for example Tom's case, section 7.3.1.1). Chapter Eight will illustrate children's expressions of randomness employing quantitative judgements, referring to the judgement of equality, the law of large numbers, the idea of uncertainty and proportional thinking.

CHAPTER EIGHT

Quantitative Ideas of Randomness

8.1 Overview

This chapter concentrates on the children's quantitative ideas of randomness, focusing on their employment of quantitative judgements. This chapter is also part of the Phase 2, learning investigation phase. The chapter is divided into five parts. The first describes children's judgements about *equality*; the second refers to ideas about infinity and the *law of large numbers*; the third describes ideas about *possibility*; and the fourth section deals with *proportional thinking*. The last part of this chapter presents a summary of the chapter, some further analysis and provisional findings of the data presented in the previous four parts

8.2 Judgement of equality

The children's judgement of final equality in their games was expressed when the children had to construct a fair probabilistic environment. The analysis of the data presented here was based on code D4: Judgement of equality in fairness (see in section 4.3.5), which refers to children's expressions for judging equality in their game.

Paul: '…It (the space kid) is above the yellow line… Ah! Something happens now! 40 – 43. It's a small difference. …Oh… look almost equal! Now, more blues…That's ok! Someone might get more, but they will be equal. Our space kid will be **near** the yellow line.'

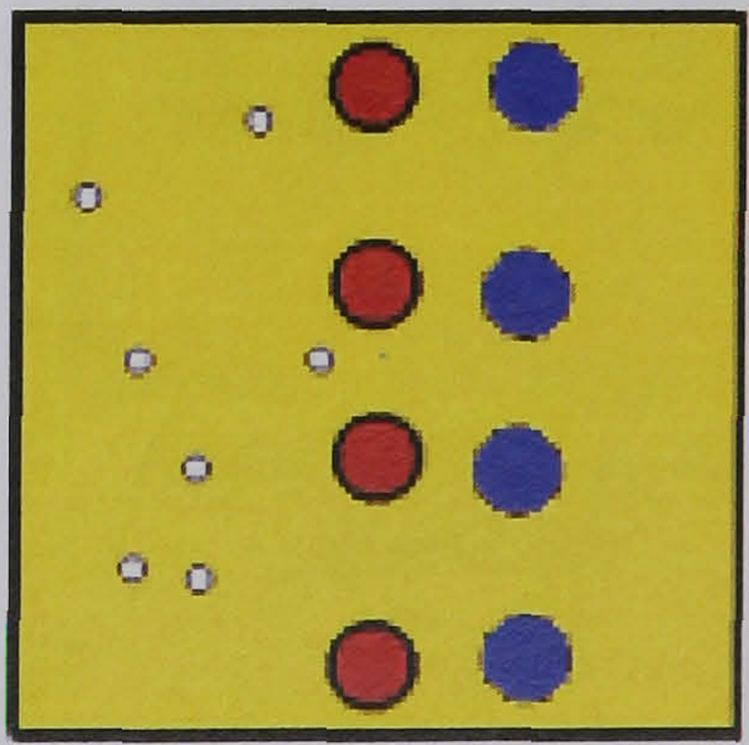


Figure 8.1: Paul's fair construction

Paul (6 10/12 year-old boy), above, constructed a fair sample space and he also used multiple white balls. As he said, the sample space worked 'ok', even though the space kid ended up a little above the yellow line (the score was 40-43). For Paul, the judgement of equality was based on having the space kid *near* the yellow line. He used the global outcome, the movement of the space kid, to judge what is equal and he did not look at having exactly the same numbers on the scorers. As he explained 'someone might get more, but they will be equal'. Although there seems to be a contradiction between the expressions 'having more' and 'be equal', this suggests that for Paul the absolute difference between the scorers did not play the central role in judging his construction. A similar episode happened with Tom (7 year-old boy). Tom explains that 'equality' means when the scores are *almost* the same (like 84-86, 99-98, 228-222). Lucy (7 8/12 year-old girl) also described that what counts in a fair game is not to have a big difference between the two scorers. As she said 'I can make something… I will do something not to be a big difference between them. To be near…' Lucy's idea agrees with Tom's one, and explained that what does matter on equality is to have only a small difference between the two numbers of the scorers.

In the extracts above, the children used the scorers to explain the equality of the outcomes. On the other hand, most of the children used the space kid to judge this equality. Jane (6 7/12 year-old girl) explained 'It (the game) is going to be fair when it (the space kid) will be in the middle, may be a little up or little down, depends on this white ball'. It seems that Jane here used the word 'little' in reference to the movement of the space kid to express the equivalent meaning of what Lucy meant by using the word 'almost' for the equality of numbers. Helen (7 6/12 year-old girl) also used the space kid to describe the equality between the two teams

Helen: I don't know… Maybe it will stay near the yellow line.

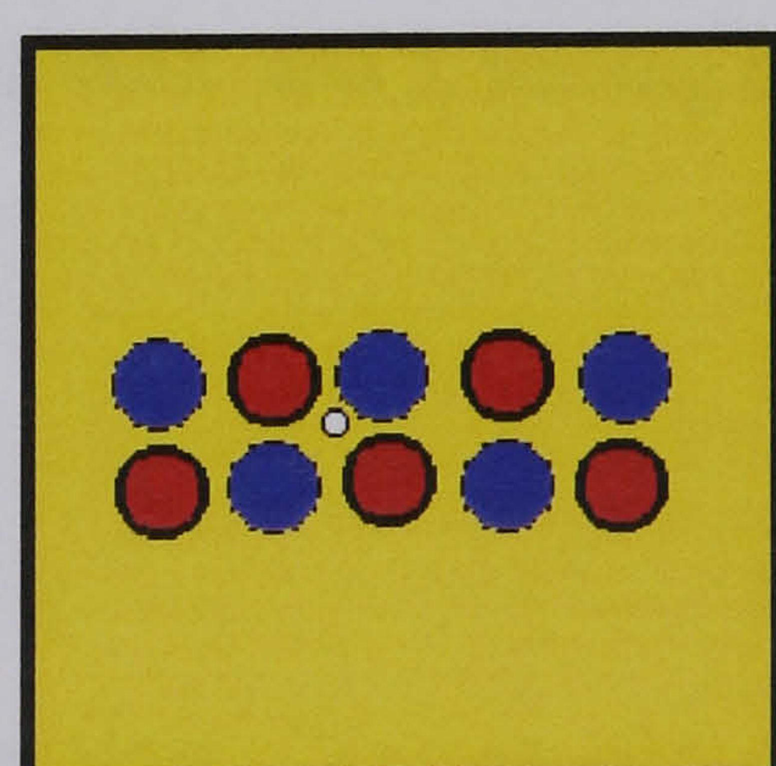


Figure 8.2: Helen's fair construction

She starts the game.

Helen: It remains near the yellow line. It goes down or up and when it comes to reach the blue planet it goes up and when it comes to reach the red planet it moves down and then it moves to the yellow line.

In this episode, Helen used the word 'near' to express her judgement about equality. She seems to understand equality in fairness by using the global outcome in the game, the space kid, and she judged her construction by the fact whether or not the space kid was near the yellow line. As she said in her vivid description, the space kid went down or up and when it moved away from the yellow line, got near to one of the two planets. Another criterion for children judging their construction was to generate 'big numbers'. This leads us to the next section, which describes children's constructions in the game for intuitively achieving the law of large numbers.

8.3 The law of large numbers

The children expressed in a number of different ways the idea of having a big number of outcomes and then judge their construction. From a mathematical point of view, these expressions point towards the situated abstraction (see section 2.3.2) of the law of large numbers[26] in terms of the mechanism of randomness or the state of the objects in the game. The analysis of the data presented here was based on codes D2: Changing the mechanism and D6: Infinity (see Chapter 4, section 4.3.5), which refer to the changes that children did on the structure of the computer game and also on their expressions of infinity. Their constructions can be categorised into: 1. increasing the speed of the white ball (expressed by 7/23 children), 2. adding more coloured balls in the sample space/distribution (expressed by 16/23 children), 3. adding more white balls (expressed by 21/23 children), 4. making the size of the white ball(s) bigger (expressed by 5/23 children), and 5. leaving the game to work for longer time (expressed by 23/23 children).

8.3.1 Increasing the speed of the white ball

The idea of changing the speed of the white ball occurred when children had made their fair construction, but could not get their desired result. Paul explained:

Paul: Let's see… Oh! We have more blue scores. It moved down. Oh…I will change the speed of the white balls. I won't watch the numbers. It will move too fast!
He takes the star and changes the speed of the white balls.

[26] This idea will be looked at more globally in Chapter Nine, section 9.4.6.

Paul made the white ball move faster than before in order to demonstrate that his construction worked, since he believed that, in the long term, his construction was working. The same idea came from Anne:

Researcher:	Let's think now! What can you do to get the equal scores?
Anne:	Make it faster!
R:	Which one?
A:	The white ball!

These examples show that the children were not thinking of making changes in the lottery machine construction, but they wanted to have 'bigger numbers' in the game. This is also an expression of a belief that their construction was working for big numbers. The next section describes where children added more coloured balls to achieve a 'bigger result' in order to judge if their construction was working or not.

8.3.2 Adding more coloured balls

Getting more points quickly was also expressed by adding more balls inside the sample space. For example, Simon (7 10/12 year-old boy) added more balls to his construction.

Simon:	It moved up now...equal numbers! Oh! Now it moved down.... Let's see if it moves up. You know something. I will add some more balls.
	He stops the game.
S:	I will put these balls together...to communicate (*he laughs*).
Researcher:	What do you think will happen?
S:	We are going to have a better result.

Simon did not change the basic idea of his construction, but he expected that by adding balls to get bigger numbers his idea would work in the long-term. His action to increase the overall number of balls may be indicative of an implicit application of the law of large numbers. Adding more balls was a strategy that children used very often for constructing a fair sample space. This strategy was also very often combined with other strategies in which children generally expressed the idea of having more trials. Sixteen children also accelerated the results in their game by adding more bouncing white balls before they judge their construction whether was working.

8.3.3 Adding more white balls

Adding more white balls was a strategy used by nineteen out of twenty-three children in order to make their construction work for 'bigger' numbers. Fiona's (7 year-old girl) attempt to get bigger scores was to copy more white balls and make them to touch more easily the coloured balls.

Fiona: It still doesn't work! I think I have to make another change to the balls. Another ball.

Researcher: Will it work with this change?

Fiona: Yeah…the white balls move around and touch all these balls. Ok… and another thing (*she copies more white balls*)…that makes it work! Wand…wand…right! Let's try it on.

She starts the game.

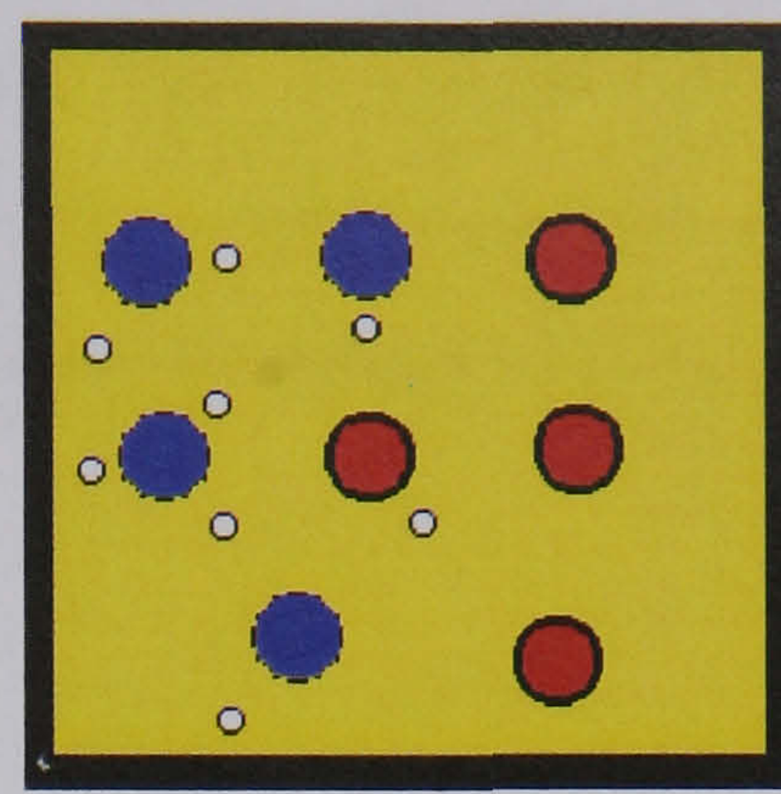


Figure 8.3: Fiona's construction to get 'more points'

Fiona added more coloured balls in her construction and she also added more white balls. Fiona's action can be seen as a situated abstraction of the idea that bigger outcomes judge better her construction. As she said, her construction would work better by having more white balls that would make the scorers move more quickly. Fiona appears to recognise that her construction would be good if it would give a proper result in the long-term. She decided not to change anything in the structure of the coloured balls, but to judge her construction by generating bigger numbers and watching the global outcomes. Another change to the bouncing ball for getting more points was to change its size as the next section describes.

8.3.4 Making the size of the white ball bigger

Making the size of the white ball bigger was an attempt to make the scorers to work quickly on the screen. This was what Mathew did (a 7 year-old boy) when he wanted to get 'many points'.

Mathew: …I will do something else. (*He stops the game*). I will construct two white big balls. I will copy some more red balls, five as the blue ones.

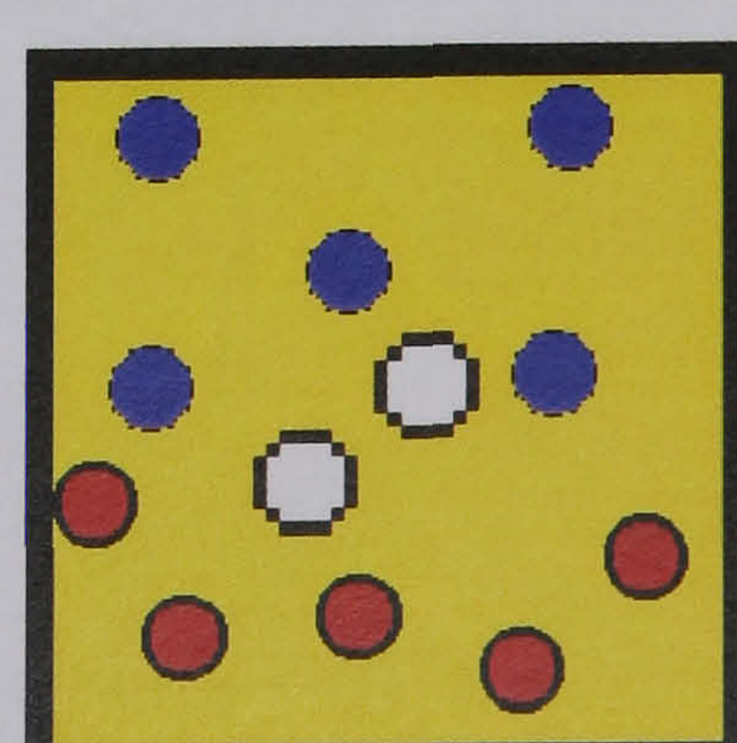


Figure 8.4: Mathew's construction to get 'more points'

Researcher: Now, we have 5 reds and 5 blues, how many points will they get?

Mathew: Many points…they will get equal points.

Mathew made changes to get bigger number scores by adding more red and blue balls, adding another white ball and making the size of the white balls bigger. Mathew's statement implies that he is attempting to get equal numbers and there is a need to get large numbers in order to achieve this. The following section describes another idea for getting bigger numbers in the game, not by making any changes in the mechanism of the game, but leaving the game to run longer.

8.3.5 Leaving the game running longer

Eleven children out of the twenty-three expressed the importance of time in their game and they expressed it in different ways. They first needed to see their construction working for a while and then judge it. Helen expressed her idea of giving more time to her construction by not stopping the game and keeping a watch on it.

Researcher: What will happen?

Helen: The blues get more points…. Now they are equal. **We need to leave it for a while.**

After a while she added:

Helen: Yes…here it is! It keeps going… It doesn't touch the planets.

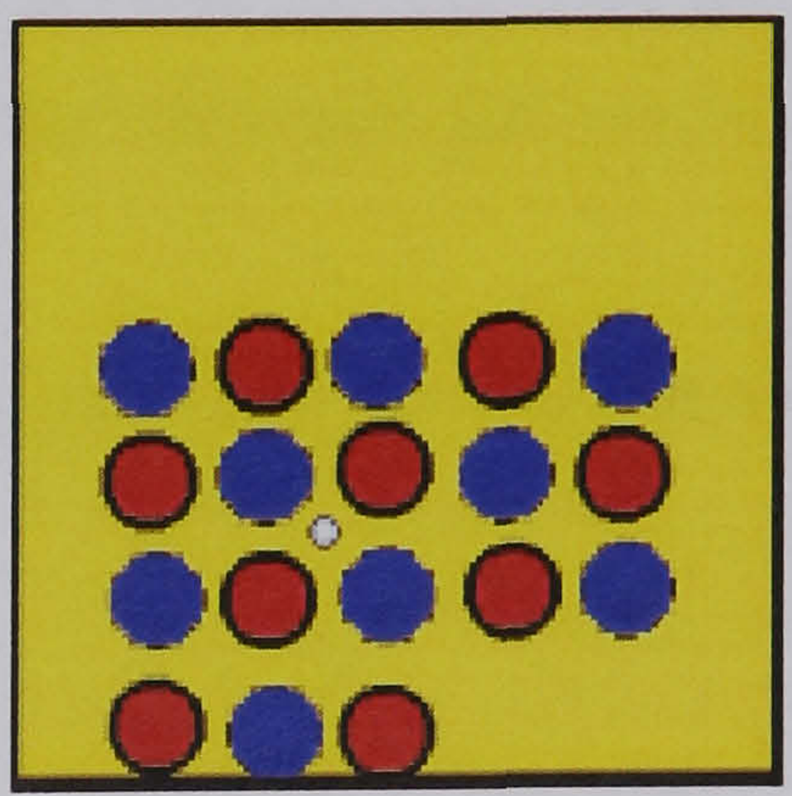


Figure 8.5: Helen's idea for getting large numbers

Researcher: Will we get 1000?

H: If it keeps moving up and down all the time and not go too (much) up then we will get 1000… We can keep it going the whole night! Ah… 700!

For Helen, leaving the game to work for a 'whole night' without stopping was a criterion to judge whether her construction was representing fairness. If the construction would work for the whole night (if the space kid would not touch one of the two planets and stop the game) that meant the construction was fair. Evidently, she used the expression of ' the whole night' to 'express' the law of large numbers in some form. This is an impressive statement, for a child to see that letting something work for a long time would produce a convergent outcome.

Orestis (7 year-old boy) expressed an idea about time by moving the two planets to the edges. As he said 'First, I will move the planets on the edges' and he was waiting for the scorers to get equal points.

Researcher: How did you arrange them?

Orestis: I mixed them up. Now, they might get equal numbers.

He starts the game.

R: Are they getting equal points?

O: **Not yet.**

Orestis expressed the idea that time is needed to get equal points. His words 'not yet' are evidence that he needed time to wait for his construction to work. He implied that his construction must be judged in the long-term and he seemed to believe that time would take care of the (short-term) inequality.

Demis (7 5/12 year-old boy) also did the same thing with the planets, giving his explanation as follows: 'Ah!... I have to move these two planets as well... I will put them down here and up there...not to touch them'. The idea was for the space kid not to touch them so quickly. Another child, Mathew, explained:

Mathew: ...But, it (the space kid) touches them. Shall I put them (the planets) a little far away from it?

Researcher: If you want to. Why?

M: I will put them here...

R: It (the space kid) won't get them there.

M: Ok! I will put them here then. To take a while to get them. I will also put a fairy there, to watch them. I will put it here.

R: Will our space kid get a planet now?

M: May be if it goes too high.

He starts the game again.

M: It will take a long time to go to planet. It will move up and down... They (the scorers) get equal points...

Mathew wanted to move the two planets in order for his game to work for a 'long time'. The two planets had been placed at the beginning, by the researcher, near the yellow line. Seeing this, Mathew realized that the space kid would touch them at once. So, he wanted to place the planets away, at the edges of the screen, in a position that it would be 'impossible' for the space kid to reach them. His strategy is to let his game run for a 'long time', so that the scorers will equalise.

As the above snapshots show these children recognised a need for leaving their construction to work longer in order to judge it. Thus, the children expressed the idea of judging their construction in the long-term with a need to have larger numbers in their outcomes, implying the need of the 'law of large numbers' in their game. The next section describes how children faced the idea of uncertainty in their game.

8.4 The idea of uncertainty

Twenty-one of the children often expressed uncertainty for an event to happen for sure, considering that they could not be completely sure of the result of the game. The analysis of the data presented here was based on code D7: Possibility (see section 4.3.5), which refers to children's expressions of uncertainty. Lucy expressed this possibility that everything can happen in sample space by using the words 'sometimes' and 'may be'.

Researcher: Is it going to be fair now?

Lucy: Maybe, let's see!

She starts the game.

R: So, what's the result?

L: Sometimes the red has more points…

She stops the game.

L: Ok…We have ten balls. 5-5.

R: Is our space kid going to be near the yellow line?

L: I don't know. Sometimes yes, but maybe sometimes not.

R: What do you mean?

L: Sometimes they will be equal. But, I have to start the game first.

She starts the game

Lucy expresses some 'fear' of uncertainty accompanying her predictions by using the words 'sometimes', 'may be', 'I don't know'. The only thing that would convince her was to try her game out. The same happened with George (6 8/12 year-old boy) who wanted to start the game first before he expressed his thoughts about what would happen in the game.

Researcher: How many points will we get?

George: We have to start the game. Nobody knows. We have to turn the game on.

...

Researcher: ...Ok... Let me ask you something. If we have another blue and another red ball, how will our space kids (*he had copy another space kid*) move?

George: Nobody knows about it. You have to start the game in order to know. I couldn't know.

R: What might they do?

G: It may touch the red ball and they will move up.

R: What about the scorers here?

G: May be, but may be the red will get more, may be the blue will get more. Let me start the game.

He starts the game.

George understands that everything is possible in the sample space, that he can only make predictions for the result, but not to be sure about it until he starts the game and finds out. He also seems to realise that in the short-term, he may not have the same results even if he kept the same structure of the lottery machine.

Demis (7 5/12 year-old boy) also admitted that he was not able to know the exact final result of a trial.

Researcher: Will our space kid move to a planet?

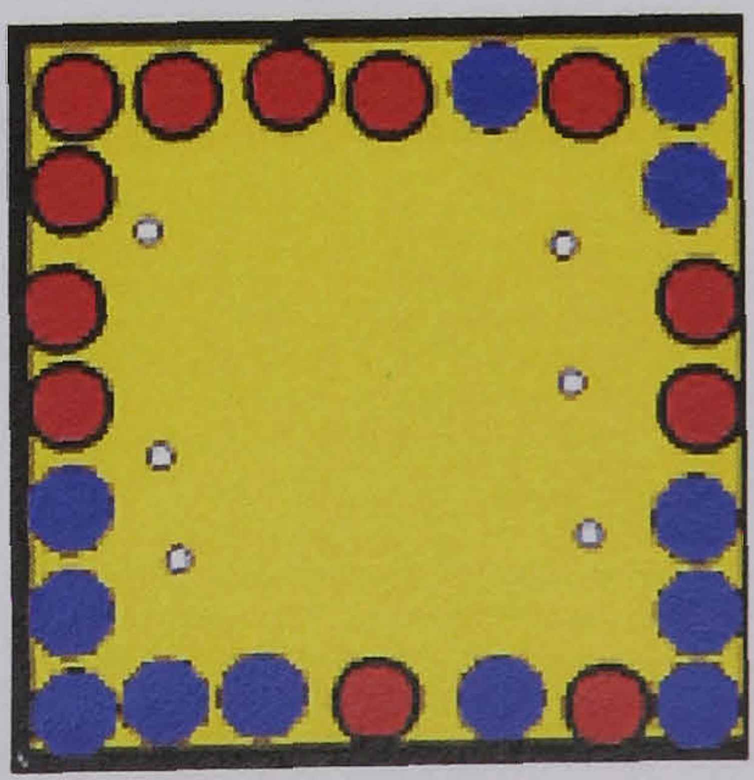


Figure 8.6: Demis' construction for expressing possibility

Demis: I don't know.... I never know **exactly** what number they (the scorers) will get.

Demis expresses the fact that he never knows exactly what number the balls will get. It seems that he uses the word 'exactly' to refer to the local events of the game. Even though he made his construction in order the space kid not to touch any planet, he also understood that this might not work as he wanted. In the second quotation he asserts that the score will reach 500, and then expresses his fear that his prediction might not happen. He also refers to needing time to get his result, tacitly implying a 'law of large numbers', as analysed in the previous section.

Rachel (7 3/12 year-old girl) expressed her belief that everything is possible, even when the probability of an event is small.

Researcher: Ok! The blues will get 6 points, how many points will the red get?

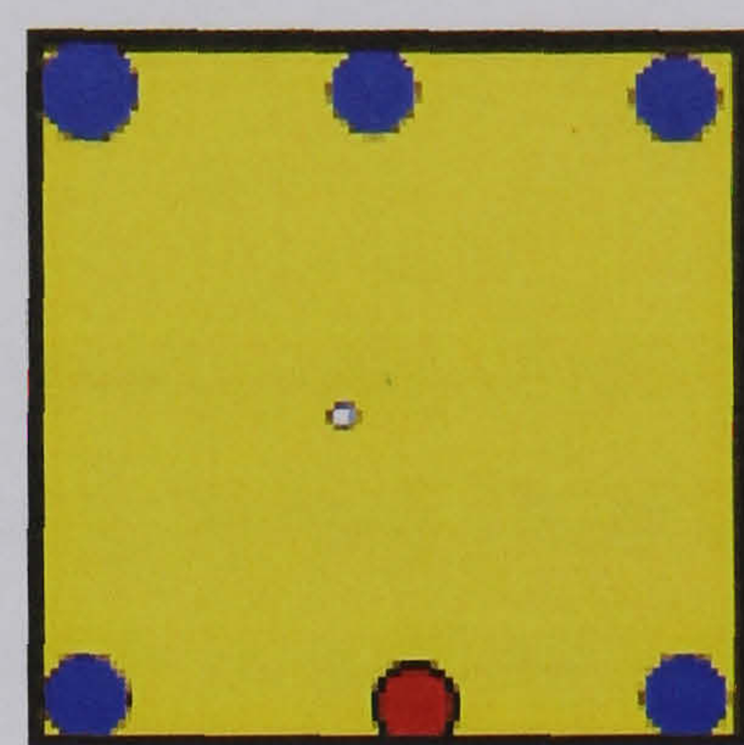


Figure 8.7: Rachel's construction to express possibility

Rachel: 1 point. You see, we have 6 balls…Hm… you know something…the blues will get 5 points and the red one.

Researcher: Why?

Rachel: The white ball will move like this and then like this and here…so it might get this ball here and then to move to the red like this. It is possible, you know. We could also get 6 blues and 3 reds, may be…it is possible.

Researcher: Which one is the most likely to happen?

Rachel: The most likely will be the blues to get more points.

Rachel predicts that it is more likely for the blue to get the most points, but she also explains that it is possible for red to get more points than one. She was trying to guess how many points each colour would get, by thinking about the possible movements of the white ball in order to touch the red ball. As she said, it is possible to get 6 points for blue and 1 point for red, but also possible to get 6 points for blue and 3 points for red. This snapshot also shows Rachel's attempt to express a proportion for the outcomes, which leads us to the next section describing children's ideas about proportional thinking.

8.5 Proportional Thinking

The analysis of the data presented here was based on code D5: Proportional Thinking (see section 4.3.5), which refers to the way children tried to express proportionality in their games. The children's ideas about proportional thinking can be divided into three categories: a. equality of two events, b. double points, and c. probability of an event.

8.5.1 Equality of two events

Almost all of the children (22/23) seemed to understand the idea of equality of two events, that is, the idea of 1/1: 2/2: ...:8/8:...:n/n. In the following episode, Anthony (5 10/12 year-old boy) expresses this idea by comparing a previous construction of eight balls for each colour with a construction having three balls for each colour.

Researcher: Let me take away some balls and I will leave three reds and three blues...What do you think will happen?

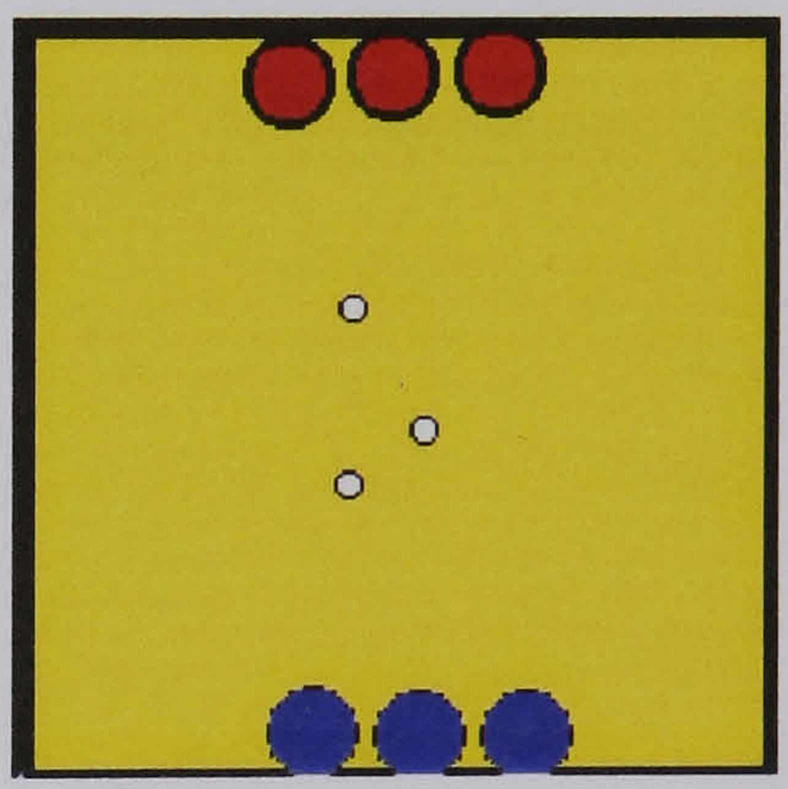


Figure 8.8: The construction for Anthony to express equality of two events

Anthony: Maybe we will get equal points...they are 3-3 balls so we are going to have equal numbers...before we had 8 and 8 balls and we had equal numbers...so, now it will be the same. Shall I start the game?

Anthony expressed the idea that having 3-3 balls or having 8-8 balls is the same and they will bring the same global result, the space kid to remain on the yellow line. Although this seems obvious, it is remarkable how Anthony, implicitly, sees the ratio 3/3 equals with 8/8. Similarly, Victoria (6 6/12 year old girl) explained:

Victoria: It (the white ball) might move here and here and it might move here. I didn't know where it was moving but it moved in different places. To have equal points we need to have 1-1, 2-2, 5-5.

Researcher: 25-?

Victoria: 25, but the yellow square doesn't have space for 25 balls.

Victoria also understood that for equality of an event it is needed to have the same number of balls in each team, when the other variables of the balls, like size and arrangement are the same. As discussed in the previous chapter (on construction of fairness and unfairness), the children found ways to have equality between the two colours without explicitly counting the balls, for example by making a pattern -they did not mind about how many balls they had inside the yellow square, as long they had an equal number. This is further evidence of the children's understanding of equality between different constructions. Although, most of the time they preferred to add balls in the sample space rather than take balls away. The next section discusses how children expressed the idea of 'doubling' the results in their game.

8.5.2 Double points

Some children expressed a meaning for 'doubling' in their games. For example, Chris (7 8/12 year-old boy) seemed to make ratios of numbers.

Researcher: Can you make something in order for the red to get double the points of the blues?

Chris: You mean two times?

R: Yes…

C: We have 8 points (the blue scorer showed from the previous trial the number 8), so the red should have 16…So, let's have 2 red and 1 blue!

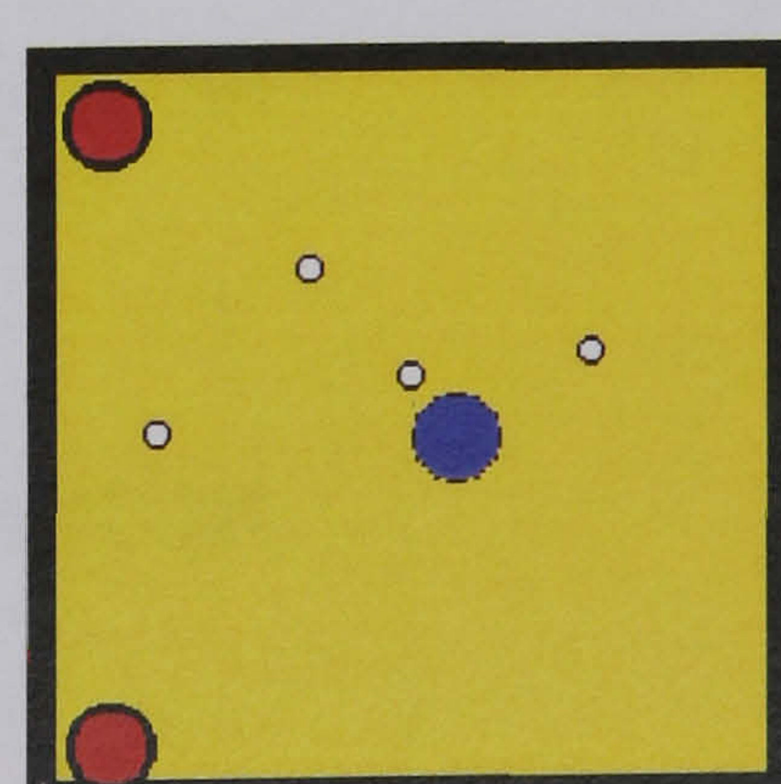


Figure 8.9: Chris' construction to get 'double points'

R: Will it get double points?

C: May be this will get 10 and this 14… I don't know… may be more! I don't know.

Chris here was looking at the previous number in the blue scorer and thinking how many points should the red get when the blue scorer was showing 8 points, and after that he simplified this to two reds and one blue ball. Finally, he constructed his sample space, by placing the blue ball in the middle. Although he made all these correct adjustments, he finally expressed his feeling that he could not be sure of the result and the scorers might get different numbers than the ones that he predicted.

Tom also (7 year old boy) gave an explanation of doubling:

Researcher: Now, there are two reds and four blues. How many points will the reds and the blues get?

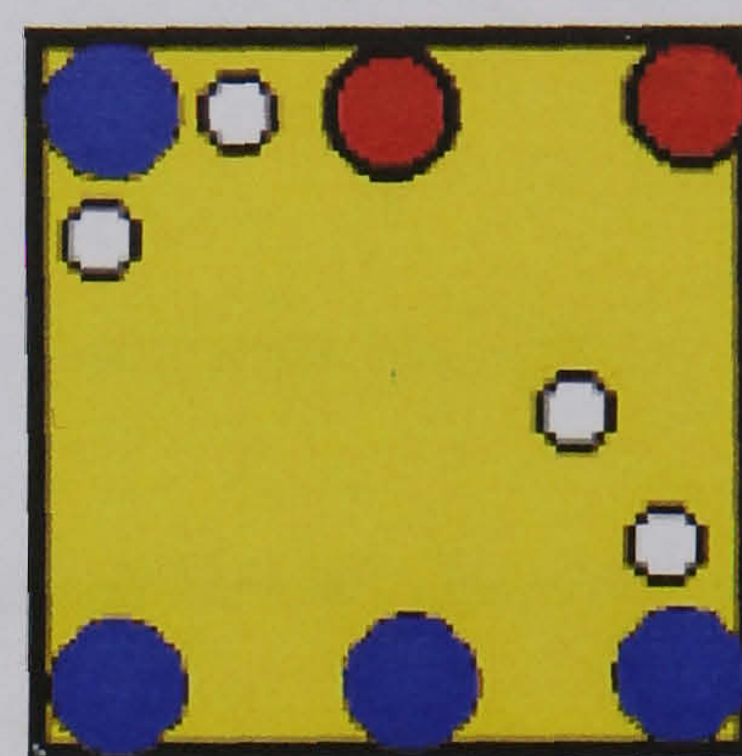


Figure 8.10: The construction for Tom to express 'double points'

Tom: 20 the reds and 40 the blues.

R: Why?

T: Because these are more.

R: If the reds get 30 how many points will the blues get?

T: I don't know. I understand that the blues will get 40 and the reds 20 points… I am not sure, because my computer didn't do this.

R: How do you understand this?

T: We have 4 balls there and 2 balls here that means 10 and 10 makes 20.

R: Why does it make 20?

T: If you have two balls and when this is here and this here again we are going to have two points.

R: If we are going to get 40 points for the reds?

T: These might get 20 points.

R: How many blues and how many reds?

T: 4 blues and 2 reds.

R: If I get 40 blues?

T: 20 reds.

R: If I get 80 reds?

T: 30…

Tom first explains how one team would get 40 and the other 20 points and he also expresses the possibility of the reds to get 10 and the blues 20. For him it was easy to see that 10 and 10 makes 20 and 20 and 20 makes 40, but it was not so obvious to find out two equal numbers that made 80 (perhaps because 80 was a too big number for him to deal with). He also understood that he could not be sure about the result. At the point that he could not find a method of dividing 30 by 2, he said that the reason of this was that his computer at home did not make this calculation!

Cathy (7 6/12 year-old girl) also chose to try out first the construction of doubles and then to justify her answer.

Researcher: Let's have now 6 balls, and the reds are two and the blues are 4. How many points do you think each colour will get?

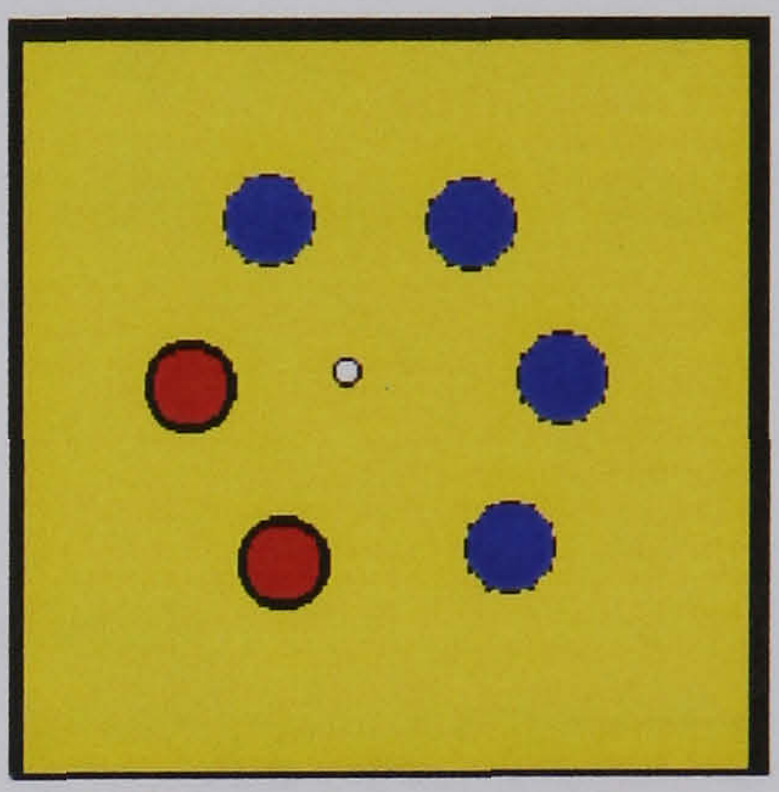


Figure 8.11: A construction by Cathy to express 'double points'

Cathy: Hm… 7 points for the reds and the blues…eh…may be 12. Can we try this out?

R: Ok!

She starts the game.

R: Let's see when the reds get 7 points how many points will the blues get?

C: Hm…Look! The reds got 8 points and the blues 16 points.

She stops the game.

R: Why did they get 16 points?

C: Because the blue balls are more than the reds. Eh…look! 8 and 8 makes 16! Two eights make 16. Double points. 2-4 so, 8-16!

Cathy started by making a guess about how to get double points, and she immediately tried it out. When the results showed the two teams had exactly double points between them, she stopped the game and made her calculations. She then compared the result with the numbers of balls and thus found the proportion between them, 2:4 = 8:16.

In the case of Paul (6 10/12 year-old boy) there was an expression of difficulty about the concept of 'twice' when he had a small number of balls and a bigger one.

Researcher:	If we want the red scorer to get twice as much as the blue scorer, how many balls do we need?
Paul:	We need two red balls and one blue.
R:	If we have four red balls how many blues do we need?
P:	3 blues… Let's see!

It was easy for Paul to express the concept of 'twice as much' by using the proportion 2:1, but it seemed that he could not manage with bigger numbers than this. Paul's expressions imply an understanding of 1-1 correspondence, he wanted 1 point for the blue so he would put 1 blue ball and 2 points for the red, and so started place 2 red balls inside his lottery machine.

Another explanation of doubling came from Demis (7 5/12 year-old boy).

Researcher:	Let's say now that we have 4 reds and 2 blues. How many points will the red get and how many the blue ones?

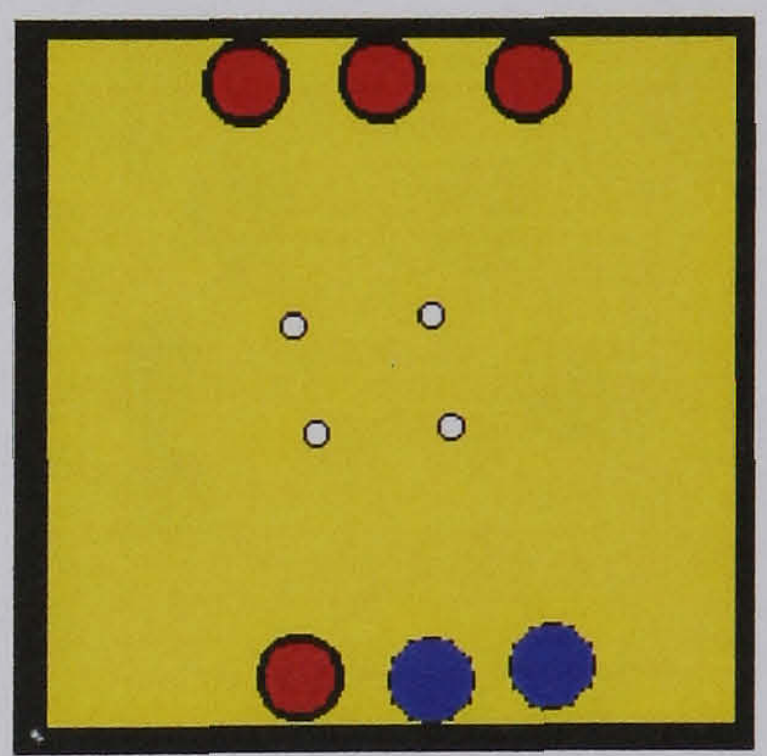


Figure 8.12: A construction by Demis to express 'double points'

Demis:	I think the blues will get 20 points and 40 the reds.
R:	Why do you think that?
D:	Because it is one more.
R:	Hm…it's one more…what do you mean?
D:	One more and it gets the reds…we need two more balls, one to add for the blues and one to take away from the reds.

R: Why will they get 40 and 20?

D: Because each one let's say gets 10 points. The red points get 40, because they are 4 balls and the blue points get 20 because they are two balls.

R: If I want the reds to get double points from the blues what else will you do?

D: **Leave it as it is.** We don't need to do something more. If this and this (the red balls) don't exist then they will be equal, so this is double than this one.

Demis was happy with this construction and he expressed a belief that it would work for getting double points for the red balls. As he described, if there were not two more red balls, then the game would have equal scores. Demis here multiplied by 10 each ball in the lottery machine. It seemed to be easy for him to see the relation between 2:4 and 20:40. He also seemed to understand doubling in terms of taking away and adding balls from one team to the other. In the following snapshot he expressed his ideas for 'ten times more'.

Researcher: Ah… Can you make something in order for the reds to get ten times more points than the blues?

Demis: Eh… ten times more? I have to copy balls. 1,2,3,4,… I put 11 reds and 2 blues…

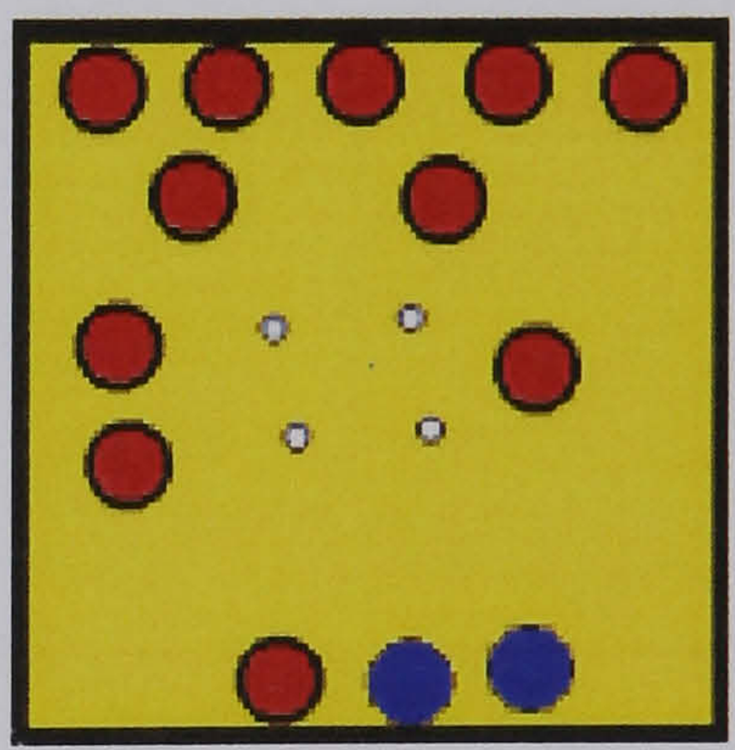


Figure 8.13: Demis' construction for 'tens time more'

R: Ok… So, when the blues get 10 points, how many points will the red balls get?

D: I think the reds will get 200… Let's see…

He starts the game.

D: Oops… I don't know, it doesn't work!

As Demis counted 10 points for each ball inside, he was asked then to predict the opposite and make a sample space with the reds to get 10 times more than the blues. This modification was difficult for Demis. He seemed to make just a guess for creating a sample space, which he eventually realised did not work. This evidence shows that also for Demis doubling is not a ratio. Demis' expressions imply an easy understanding of 'doubling' points and a difficulty of understanding the proportion for tens.

George's construction for doubling went as follows:

George:	I will put them like this. If it moves there it will touch the red and if it moves there it will touch the blue. We need another ball.

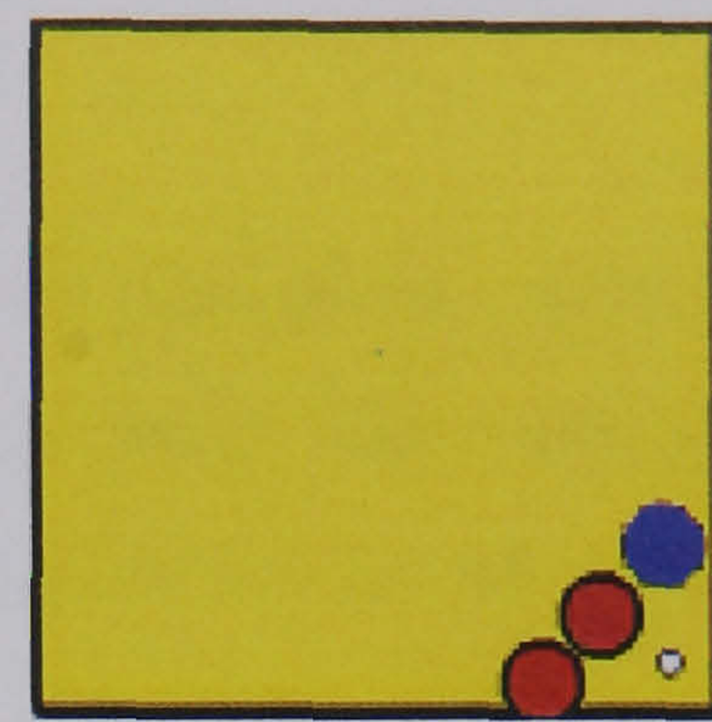


Figure 8.14: George's construction for doubling

Researcher:	Which colour will get more points now?
G:	May be the reds because they are two.
R:	Where will our space kid move?
G:	Upwards.
	He starts the game.
G:	Now it gets the blue, it gets both…now the red.
R:	How many points will the red have?
G:	It has 10 points and the blue has 5 points. You see, two times 5 makes ten, it needs another 5. We need another blue to get 10. I will stop the game. I have a very good idea. I will make something else… you see in a while. I will make a shape.

George started thinking at the scale of 1 and 2 points. But, after he started the game, he realized that this construction worked for 'multiplying by two', so it would work for any double he wanted. As he explained, two times 5 makes 10 and that worked in the same way as two times 1 makes 2. Thus, George did not initially see the relation between the numbers, but after he started the game he began to think about multiplying the numbers. There is evidence here of the game mediating George's thinking on how he can double the points.

This section has shown that the children expressed their thinking about doubling based on the part-part relationship. Although their ideas on doubling the outcomes seemed to be expressed easily and they used the medium to confirm their thought, it seemed that they did not connect doubling with ratio. The next section describes children's expressions on the idea of probability of an event and how they tried to predict the result of their construction.

8.5.3 Probability of an event

Concerning the probability of an event, twenty-one of the children expressed an outcome not as a ratio, but by saying which colour is going to get more or less points. They seemed to prefer not to estimate the scores for the two colours and they preferred to start the game first. Zeta (6 4/12 year-old girl) made the following comments.

Researcher: Let me ask you something. If I put 7 blue balls and 1 red what is going to happen?

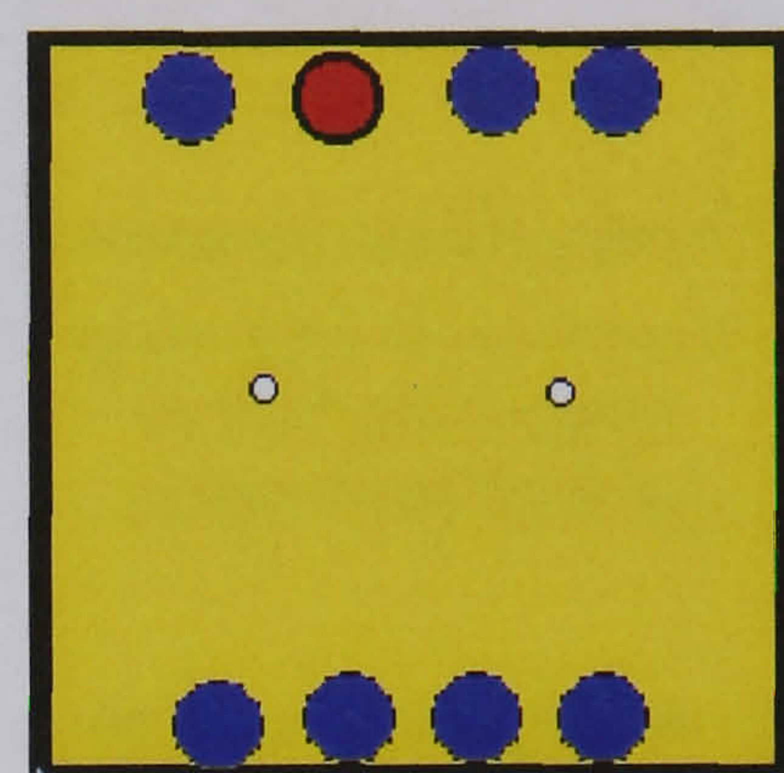


Figure 8.15: A construction for Zeta to express the probability of an event

Zeta: The white balls will touch the blue balls most.

R: How many points will the blue get and how many the reds?

Z: I don't know. Let me start the game.

She starts the game.

For Zeta it was obvious that blues were going to get more points, but she did not make any calculations to predict the scores. Irene (7 6/12 year-old girl) seemed to express how many points each ball would get by guessing a big number for the team that has more balls and a small number for the team that has fewer balls.

Researcher: Ok… So, they are 10 balls, 9 blues and 1 red. If the red gets 10 how many will the blue get?

Irene: 19… Because the red is one and the blues are 9 and they surround the white ball.

R: If now the white ball touches 100 times on the balls, how many times will it touch on the red and how many on the blues?

I: On the blue might touch 88 times or 100 and on the red 78…

Irene here guessed some big and small numbers, depending on which colour was more likely to win, without even thinking whether the total of these numbers was similar to the total scores she had seen. She dealt with decisions about these numbers by using the words 'more' and 'less'. Thus, she made decisions on which colour finally is going to win, but she did not seem to think about proportion in way to express a possible number result.

Nichol (7 8/12 year-old girl) also made an attempt to work with ratios.

Researcher: Let's say we have two blue balls and one red. If the red shows 10 points how many points will the red get?

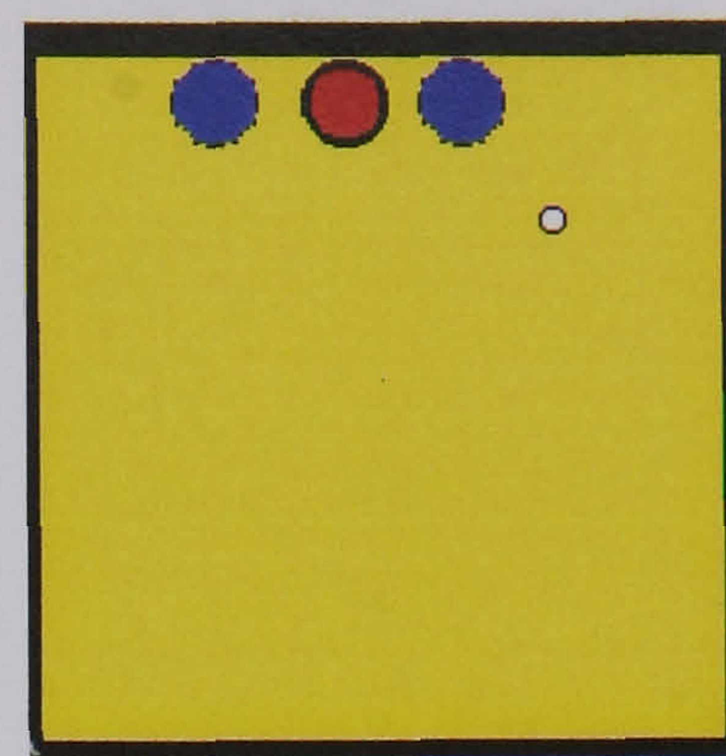


Figure 8.16: A construction by Nichol to express proportional thinking

Nichol: 10 the blues and 9 points for the reds.

R: If I put another red ball?

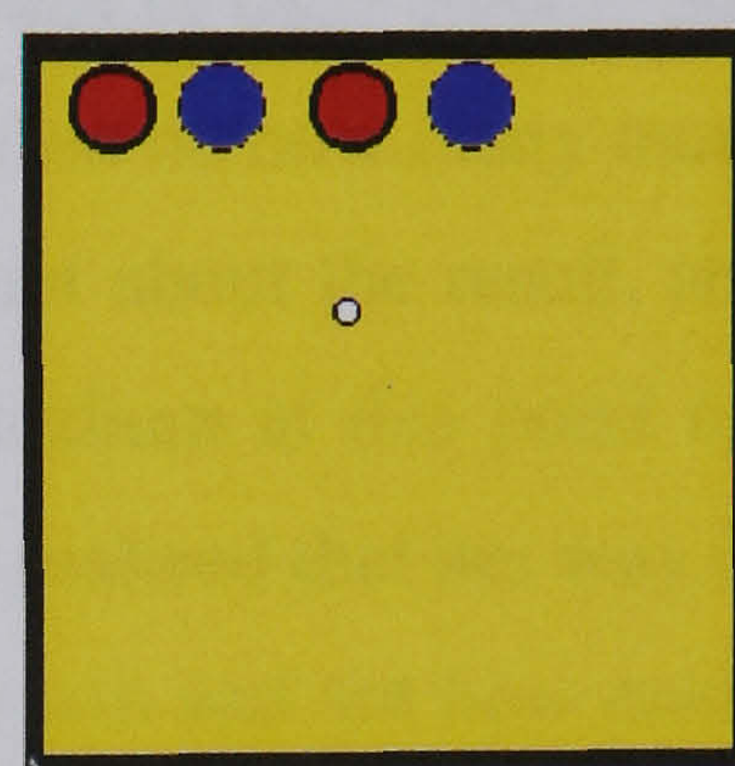


Figure 8.17: A second construction by Nichol to express ideas about ratios

N: Now we will have 10 for the blues and the red may be the same.

R: If I have another blue ball?

N: One more for the blues. Let's say 6 and 5.

R: If I copy another blue ball?

N: Then the blue will have may be 30 and 10 the reds…

R: I will put the balls like this now. 4 blues and 2 reds. What will the score be?

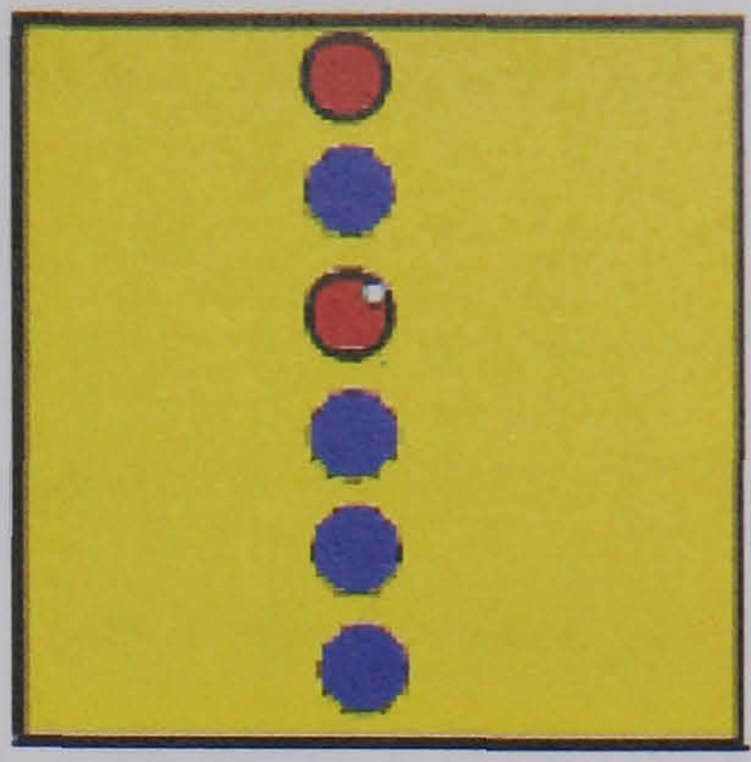


Figure 8.18: A third construction by Nichol to express proportional thinking

N: The blue will get more points.

R: How many more?

N: 4 points more.

She starts the game.

N: Hm…this might get more…It doesn't work, but the blues are winning.

Nichol here reveals some proportional thinking. She was adding and taking away from the score of the one team the difference between the numbers of the balls of each colour. She knew which team was going to win, but her predicted result was based on how many more balls one team had than the other. As she did not express any doubt about the result, she did not have the feeling to try her construction out and use the medium at this point for changing her idea. At the end, when she tried her constructions and realised that her way of thinking did not work, she predicted only which team was going to win and not how many points each might get. George tried also to give an explanation of proportion.

R: If we have now two reds and three blues?

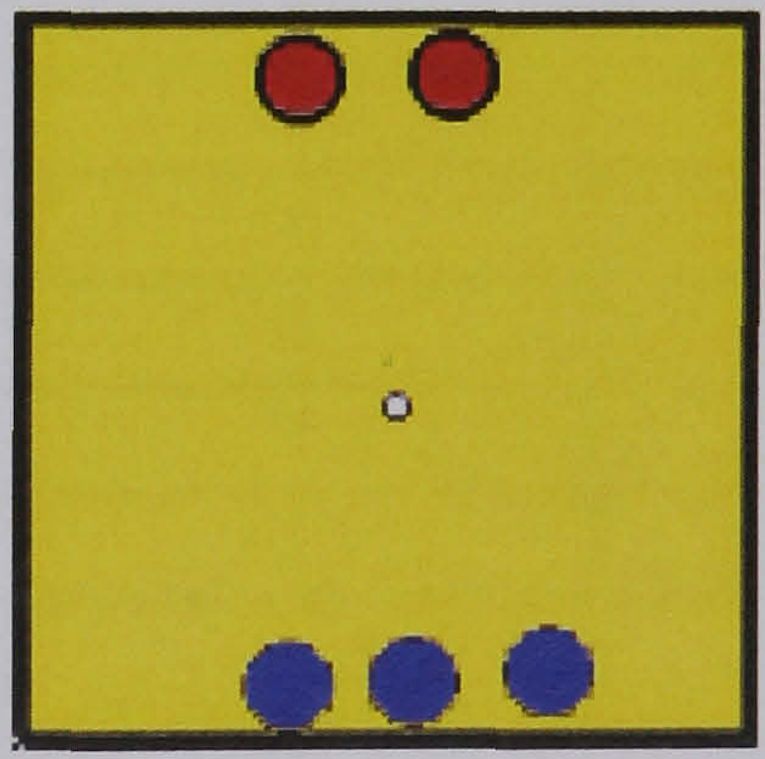


Figure 8.19: A construction by George to express ratios

G: Hm… The only thing that we can do is to start the game and find out.

R: What is your guess?

G: The blues will have more because three is more than two. You see! You know, these (the blues) will get (he is thinking by working with his fingers) 6 and the reds might get 4. You know 2 and 2 makes 4 and 3 and 3 makes 6 (*He shows me by his fingers*). I will start the game.

He starts the game.

G: Wow… It got 2… We got different points, but the blues got more.

George here doubled the number of balls, of both colours. He used his fingers as a representation to explain his thinking, and how he reached the conclusion that 2:3 = 4:6. George showed that he did not only work with a 1-1 correspondence, but he found also a way of doubling the numbers of each colour and comparing them. In the end, when he started the game, he got different numbers from the ones that he predicted (although they had the ratio of 1:2), he expressed only the idea that the blues would get more points than the reds. Table 8.1 shows which category each child used.

			Categories of 'proportional thinking'			
Name[27]	**Age**	**Sex**	**Equality of two events**	**Double points**	**Probability of an event**	
					More/Less points	**Ratio**
Anne	6:10	F	√	√	√	√
Anthony	5:10	M	√		√	
Brian	6:6	M	√		√	
Cathy	7:6	F	√	√	√	
Chris	7:8	M	√	√	√	
Demis	7:5	M	√	√	√	
Fiona	7	F	√		√	
George	6:8	M	√	√		√
Helen	7:6	F	√		√	
Irene	7:6	F	√		√	
Jane	6:7	F	√	√	√	
John	6:10	M	√		√	
Karen	7:3	F	√		√	
Lucy	7:8	F	√	√	√	
Mathew	7	M	√		√	
Nichol	7:8	F			√	
Orestis	7	M	√		√	
Paul	6:10	M	√	√	√	
Rachel	7:3	F	√	√	√	√
Simon	7:10	M	√		√	
Tom	7	M	√	√	√	
Victoria	6:6	F	√		√	
Zeta	6:4	F	√		√	
Number of children (out of 23)			22	10	22	3

Table 8.1: The total number of children expressing different categories of proportional thinking

27 All names of the children are pseudonyms.

Table 8.1 shows that 22/23 children expressed the idea of equality of two events to occur. Also, twenty-two out of twenty three children expressed the probability of an event to occur by referring to which colour would take the more and less points. It was also note worthy that although ten of the children double the points of each team for expressing the 'twice as much' idea, only three of them expressed a belief that this 'double' strategy could be generalised. Table 8.1 also shows that George is the only child who did not express the probability of an event by giving a 'qualitative judgement' (more/less points), but he tried to express probability of an event only in a ratio. I remind to the reader that George was the only child who did not construct any symmetrical strategies for expressing fairness in his game (see Table 7.1) and he only expressed unfairness with a spatial arrangement (see Table 7.2). The table also shows that the sex and the age of the children did not influence their decisions of constructing unfairness in their game.

8.6 Summary of Chapter Eight and provisional findings

The above episodes have described how the twenty-three children of the study expressed randomness by employing quantitative judgements or by making some calculations. Mainly, these judgements were involved in the children's understanding of equality in the game and their understanding of the need to have big numbers in the game (an intuitive form of the law of the large numbers). Children seemed also to express how they understood uncertainty in their constructions, and some primitive ideas about proportional thinking. The episodes of Chapter Eight lead to the following analytical points, and which will be analysed more globally in the following chapter.

Provisional finding 8.1: The children of the study tended not to use the absolute differences in the outcome scores to judge equality in fairness. Instead, they used the global outcomes of the game and made judgements based on the small differences between the different experiments (see for example Helen in section 8.2). This judgement of equality is evidence of children's making connections between local and global events, and how they shifted their thinking from the local experiments to the global events in their game. Although it was easy to read off the exact points of each team at any time, they preferred not to refer to the absolute difference but to judge equality when the two teams had a 'small variation', which could be seen from the distance that their space kid had moved from the yellow line.

Provisional finding 8.2: The children of the study often made changes in their constructions to get more outcomes. All the children left their game to run for a longer time and twenty-one of the twenty three made changes to the mechanism of the game to get more outcomes in less time. The game provided ways in which children could engage with the idea of getting 'bigger numbers' besides leaving the game work for longer. It seems that children's expressions provide evidence of situated abstractions of the law of large numbers. They seemed to understand that they would get a better result from their sample space when there are many trials. It can be said that by experiencing the game they understood that something that is unstable with a small number of outcomes becomes stable with a large number of trials. The children in this study seemed to develop an intuition about the stability of long-term trials, a shift of focus that the game promoted by looking at the aggregate outcomes of any construction.

Provisional finding 8.3: Many constructions of the lottery machine had the probability of getting the one event to be very small. Even then, the children expressed their understanding that everything is possible. They seemed to realise that extreme variability is also a possibility to happen (see example of Rachel in section 8.4). Their predictions were expressed with an uncertainty.

Provisional finding 8.4: All the children seemed to understand that 1/1:2/2:...:n/n in a medium that enabled them to make manipulations by adding and taking away balls. They seemed to realise that equal ratios would produce an equal outcome. Children could easily change their construction by changing the number of the balls. They made a similar construction of balls by changing the number of the events in their game. This behaviour occurred when children faced the problematic situation of creating fairness in their game.

Provisional finding 8.5: Ten out of the twenty-three children developed different mechanisms for thinking about doubling based on the part-part relationship. They tried to manage with a proportion that exhibits the concept of twice as much and to use the aggregated view of the environment to see whether it worked or not. The children's expressions show that it was easier for them to express the idea of doubling with small numbers than with big ones (see example of Tom in section 8.5.2). Given this, their ideas on doubling the outcomes seemed to be expressed easily and they used the medium to confirm their thought or to develop it; however, they did not manage to work effectively with other proportional numbers like tens.

Provisional finding 8.6: Twenty-two out of twenty-three children typically expressed the probability of an event in the form of 'which colour would get more points than the other'. They did not care so much as to quantity (how many points each team might get). The medium gave to 3 out of 23 children an insight to think that there might be a 'strategy' for predicting the score (see example of George in section 8.5.3), but they could not achieve proportional thinking about probability of an event- these children tried to make some predictions about how many points each team would get, but after they started the game their focus also tended to shift to which team would win or not.

The provisional findings that have been described above, as well as the provisional findings of Chapter Six and Chapter Seven will be fleshed out in more detail and would be looked at more globally in the next chapter, 'Conclusions', in relation to the theoretical framework of the study (described in Chapter Two).

CHAPTER NINE

Conclusions

9.1 Overview

This chapter summarises the findings of the research, with reference to the two principle aims of the study. First, it describes the design aim, and summarises the findings from Phase 1: iterative design phase. Second, it describes the findings from the Phase 2: the learning investigation phase, which explains how the computer game (the lottery game) mediated children's thinking about probability (the second aim of the study). Specifically, I analyse here the situated abstractions of randomness, fairness, unfairness, equality, the law of large numbers, possibility and 'proportional' thinking. Finally, this chapter discusses some implications for the teaching of probability, based on the research, the limitations of the present study and some suggestions for further research.

9.2 Summary

The main tool for this research was a computer-based lottery game, designed to help the children to connect concrete and abstract ideas, by building situated abstractions of mathematical knowledge. Pratt's (2000) and Wilensky's (1995) constructionist paradigm for understanding probability illustrates the case that by designing tools that are specially designed for expressing randomness and chance, and encouraging learners express their ideas with them.

This study provides further evidence for working in the constructionist paradigm, dealing with young children's probabilistic constructions and expressions whilst using a computer lottery game. The lottery game was specially designed for the expression of randomness and chance. Thus, the lottery game provided a 'window' onto the idea of sample space in which children could manipulate in a concrete way the elements of the sample space. This notion of 'window' (see Noss and Hoyles, 1996) is a metaphor to describe the way in which the computer interface offers to the researcher insights into the children's meaning making as they work in the microworld.

The central aims of the study were 1. iteratively to design a computer game to afford young children (age 5½ -8) opportunities and novel ways to express and develop probabilistic ideas; and 2. to describe and analyse how the game mediated the children's expression of chance events. 'Expressiveness' is meant in the sense that one can express ideas in a concrete form while actions are carried out by interacting with a tool (see diSessa, 2000). Moreover, Noss and Hoyles (1996) state that expressive power opens windows for the learner; it affords a way to construct meanings, where meanings are expressed in actions.

The next two sections summarise the findings of the study, with reference to the two main aims, findings about the design of the game, and the analysis of how the game mediated children's expressions of probabilistic ideas.

9. 3 Aim 1: Findings from the iterative design phase

9.3.1 General background to the design process

I will first describe how the idea of designing a lottery game was developed. The lottery game was designed for the expression of ideas about randomness; the intention was to create a "dynamic tool", which goes beyond the static and discrete probabilistic environments that children experience in schools. The game was designed with a program called 'Pathways' (which developed itself as the iterations of the game development took place). Pathways gave the opportunity for the game to be based on three main principles:

a. A 'lottery machine', a visible manipulable engine for the generation of random events, represented an "executable sample space" or distribution in the game. With it, the children could directly manipulate the outcome of the game.

b. The presentation of the lottery machine in the game was geometrical/spatial and it contained balls of different colours, which made it possible for children to carry out as many events as they wished without being obliged to think about numbers and also, in the final iteration, to change the probability of an event to occur by changing the size of the balls and their location/arrangement.

c. The game was programmable in the sense that the user could examine how each object was connected by rules and how it linked in a visible way the short-term behaviour of the lottery game with the long-term behaviour. Thus, the children could visualise individual outcome as a single trial in a stochastic experiment, the totality of these outcomes gave an aggregated view of the long-term probability of the total events.

Connecting objects with rules encouraged children to examine the rules in order to explain the unpredictable and arbitrary movement of the ball. For example, when children could not predict the exact movement of the ball, they thought that looking at the rules was sufficient to discover how the ball was working.

9.3.2 Findings from the iterative design phase

The early iterations of the design process gave the researcher the opportunity to observe the characteristics of children's ideas about randomness, and how they expressed these ideas. The dominant tendency was to look for patterns, and other research (for example Konold, 1989; Pratt, 1998) has shown the same tendency. This is also a claim by Piaget and Inhelder (1975), that children before the age of 8 years old are most interested in the point of view that considers the pattern of the total number of balls and the 'effect' of each experiment on the next – this latter behaviour was also observed in my first iteration. In the first iteration, the sample space in the computer game used a linear representation of coloured balls, and this confused all the children of the first iteration who thought that the colour that they had to get should be placed on the top, which made them ignore the structure of the sample space as a whole.

The main finding from the first iteration was that the 'presentation' of the lottery machine had to change, so that the children would be encouraged to look for structure rather than patterns. The specific modifications subsequently made was that children could not only add and remove balls in the machine, but also construct a 2-dimensional arrangement of the balls. The first iteration also showed that in a probabilistic game it was important for children to have a 'continuous movement' in the window of sample space to make it easier for all children to look at the aggregated view of the distribution and the probability of an event. The 2-dimensional arrangement was intended to help children to shift the focus of their thinking away from seeking patterns. Further, the scope for manipulation of the sample space was designed to encourage children to express their thinking and construct sample space in the form they wanted.

In the final iteration, the game was not only manipulable, but also presented a continuous link between the sample space and the distribution of the outcomes, as shown in diagram 9.1.

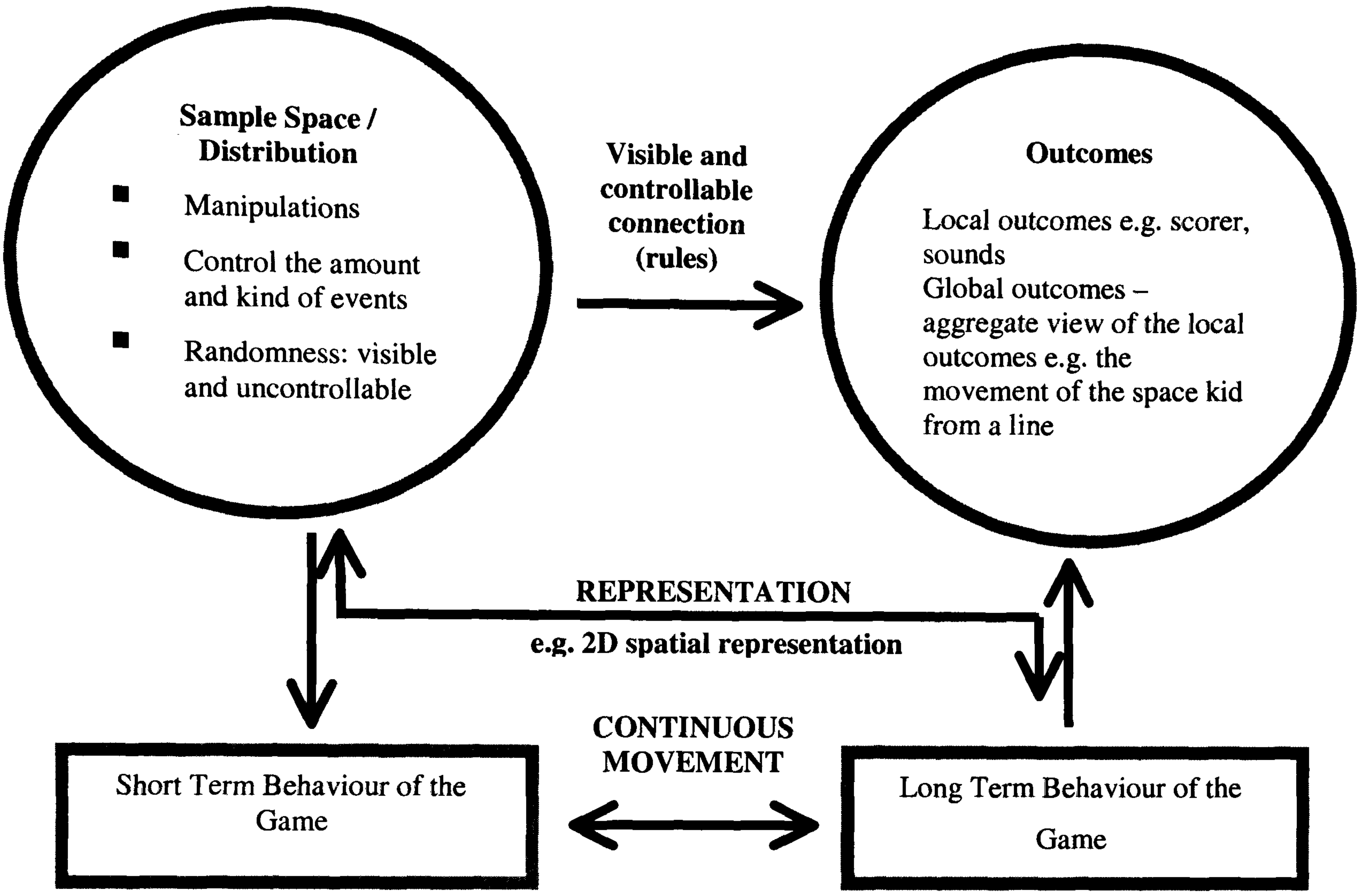


Diagram 9.1: A general design for a game on randomness

Diagram 9.1 is generalised from diagram 6.1 and presents the ingredients for the design of a game on randomness. In Diagram 6.1 the elements of the game was a. sample space, which was expressed by one child's thinking (Rachel's case) by the movement of the white ball and the outcomes, which were expressed as the movement of the space kid, the sounds and the scores. Diagram 9.1 shows in a more general way that the elements of the game are: a. a sample space and distribution that are manipulative (for example there is control over the number and kind of events), and the random behaviour exists in a visible, but uncontrollable way; b. the outcomes of the sample space and distribution are divided into local and global ones (see section 3.4.1); and c. the two dimension spatial representation provides the opportunity for the sample space and the outcomes of the game to be seen in terms of both short term and long term behaviour. In the diagram, the two circles show the elements of the game whilst the two rectangles show how the elements can be seen in terms of local and global events.

In the general design, the connection of the sample space and distribution with the outcomes has the crucial property of being both visible and controllable. In the case of the Pathways game this connection was achieved by using rules that could be constructed and changed by the children. Also, in the general design, the representation of the sample space and distribution could have a visible connection with the short and long term behaviours, through the two-dimensional spatial and dynamic representation of events.

A further important characteristic of the game was how it connected the short-term and long-term behaviours of the system, to provide children with corresponding local and global events. The research findings show that the continuous movement in the game did successfully link the short-term and long-term behaviours, thus discouraging children from simply looking for patterns in randomness. For example, in the case of Rachel (as shown in diagram 6.1) the two-dimensional continuous lottery machine made it possible for her to see the movement of the white ball not only as a short term behaviour of the system, but also as a long term behaviour. This played a major role for Rachel to connect local and global events, and Rachel's descriptions many times involved shifts from local to global.

It is conjectured that the components of the game, and the connections between the components helped children to connect "pieces" of the system mentally, and introduce corresponding structure into their thinking about randomness (as proposed by the theory of p-prims and schemata described in the literature review, section 2.2.1). These pieces can be seen as a first meaning about things, consisting of simple abstractions, not well developed generalised pieces of knowledge that originate from specific experiences but can be used in similar situations. There are many examples from the data where the design of the game helped children to link their p-prims for the understanding of randomness (these are described in Chapter Six). In the case of Victoria (section 6.2.2), for example, she began to realise that even if some of the times the white ball touched a place that she predicted, the white ball did not follow a path for its movement and she could not control or predict exactly its next move, and so she finally focused on the 'global' movement of the space kid instead of looking at the short term behaviour of the ball. There is evidence to make a tentative conjecture here: if the design of a game can instantiate what children's pieces of knowledge about randomness are, then working with the game can help children to connect ideas for randomness in their thinking. As p-prims are unstructured pieces of knowledge, designing a game where its components represent p-prims and the tasks require to connect these pieces together, then the children by interacting with the game and expressing their thoughts can begin to build an intuitive structure about these pieces of knowledge.

9. 4 Aim 2: The game's mediation of children's probabilistic ideas

The findings from the final iteration of this research provide insights into children's understandings of randomness (learning investigation phase). First, I will illustrate the findings from the Piaget and Inhelder experiment described in section 4.3.1. Then, I will describe the findings of the study in terms of four 'situated abstractions':

a. mediated expressions of randomness,

b. symmetric and asymmetric fairness,

c. the idea of unfairness,

d. qualitative judgements based on equal likelihood events, 'proportionality' and limits (the law of large numbers).

I introduced the notion of 'situated abstraction' in Chapter Two. Noss, Hoyles and Pozzi (2002) describe situated abstractions as a way of describing how a conceptualisation of mathematical knowledge can be simultaneously situated and abstract. The key idea is to take account of the way in which the tools available, the means of expressing the ideas, structure the way that the children think about and express those ideas.

9.4.1 Piaget and Inhelder's tilt box experiment

As part of the transition from the second to the third design iteration, it was necessary to try to generate some baseline data to look at the effects of the medium of the task on how children are able to express themselves. Thus, the twenty-three children of the study were asked to attempt Piaget and Inhelder's (1975) tilt box experiment at the beginning of each interview, to examine whether the sample of children would appear similar to that of Piaget and Inhelder, and also and to find out how different tools might mediate the children's understandings of randomness. Ojeda (1999) points out a weakness in the tilt box experiment in that some children may interpret the task as being to control the movement of the tray to obtain, after each tilting, the balls arranged in their original place.

Piaget and Inhelder reported that before the age of 8 years old the progressive mixing of the balls is either denied or thought of as too regular. In one respect, this, they argue, is simply another way of avoiding the idea of chance. The findings of the present study are in line with Piaget and Inhelder's findings, in the children interviewed found it difficult to express notions of random mixture, when they were trying to express these notions by using a tilt box as a medium. However, when the medium of expression was the computer-based game, the children constructed valuable meanings for randomness as shown in the previous chapters. These findings study suggest that Piaget and Inhelder's results cannot be generalised, independent of the medium of expression. By using a more dynamic medium for the expression of randomness, it can be concluded from this study that children of this age *can* express notions of randomness that cannot be predicted by model of stages of thinking, as Piaget and Inhelder (1975) or Jones et al (1997) have argued (cf. Chapter Two 'A review of the literature').

9.4.2 Mediated expressions of randomness

The children's attempt to understand randomness shifted in focus from looking for ways to control randomness, towards focusing on ways to control events in the sample space.

In general, all the children had a strong urge to control an unpredictable situation and to find ways to predict results. In most games that children of this age play the sample space remains hidden or implicit, and they do not access it as an idea. But in the Pathways game children access sample space, interacted with it and used it. In particular, the game led the all the children of the study to think in a direction that in order to make predictions, they should move from attempting to control the 'thing' that delivers randomness (as represented by the movement of the white ball in the lottery machine) to controlling the events in the sample space (as represented by the coloured balls in the lottery machine).

As the literature confirms (for example Ojeda, 1999; Ayres and Way, 2000) children have a tendency to try inappropriately to utilise colour patterns in order to predict outcomes in a random experiment (as iteration 1 of this study also shows), and this results in a difficulty to understand the overall random behaviour. The children often focus also on the 'object' that delivers randomness e.g. the dice or the coin. Truran and Truran (1999) describe how their subjects would normally toss dice using their hands in the belief that there is a direct human intervention on the outcome. Fischbein (1975) describes how children have the belief that random events can be controlled by the individual who triggers the events when, objectively, any such control is absent. Thus, the belief that successive outcomes of a random process are not independent is one of the most common 'misconceptions' reported about probability.

In the interviews of the present study, the children at first tried to find the ways in which randomness works by 'isolating' it, looking only at the object that delivers randomness and not at the whole random environment. However, the results from the final iteration of the study suggest that working with the microworld changed the children's understanding: they began to manipulate the events in the sample space instead of looking for patterns to control the random behaviour. This change of focus, seems to have appeared, because of the continuous movement inside the lottery machine, which made it much more difficult for children to look for patterns.

Kuzmak and Gelman (1986) concluded that very young children have limited ability to explain the nature of a random mechanism. However, the features of the lottery game in this study allowed children to visualise randomness in their game as not following any particular rules, as they could see that the arbitrary movement of the ball had no rules and therefore, did not follow any patterns. It seems that this interpretation of the ball's movement as arbitrary was a driving force for children to express their ideas of randomness. They realised in the long term that the lottery machine was a 'window' to the sample space and that the lottery machine did allow them to 'control' in a way the arbitrary movement of the ball in order to get the result they desired.

The mechanism of the game seemed to be a catalyst for building 'situated abstractions'. It seems that probability for the children of the study was connected with an attempt to control randomness. For example in Anthony's case (a five and ten months year-old boy), there is an attempt to control randomness by manipulating the events (see section 6.2.2.1). Anthony expressed this by changing the events in the sample space, which enabled him to control the random behaviour 'a little bit'. From a mathematical point of view the children were expressing, in their own way and mediated by the computational tool-based game, the idea that the probability of an event is a tool for describing and predicting unpredictable behaviour. The game helped all the children make this shift from focusing to control the thing that delivers randomness to focus on controlling events in the sample space. By assisting and developing situated abstractions for the idea of randomness, children seemed to be able to structure their intuitive understandings of probabilistic ideas. Within the changes made in the lottery machine children expressed ideas of probability. For the young children of this study, constructions in the lottery machine were an attempt to control an arbitrary behaviour and within the game this idea was an articulated abstraction for the law of probabilities.

9.4.3 Fairness

The children employed two representational forms to express fairness: symmetry and asymmetry. Since normally children connect only symmetry with fairness, the fact that they employed both symmetrical and asymmetrical forms for their constructions suggests also that they could engage with the idea of distribution in a simple way.

In order to construct a random fair environment, the children had to develop some strategies to achieve fairness. The interesting point here is that these strategies emerged from all the children's need of this study to construct fairness, starting from an unfair environment and creating constructions to make it fair, without having any options to select from a variety of pre-constructed situations. The strategies that are described in the next paragraphs emerged from their constructions while they were facing the problematic situation of the construction of fairness (expressed as what they can do in order for the space kid to remain close to the yellow line, see figure 5.4). The shift from outcomes to construction of models via the machine, suggests that the lottery game situation afforded the construction of fairness to be connected with both symmetrical and asymmetrical spatial representations. Twenty-two out of the twenty-three children tried to achieve a fair random game by employing a symmetrical representational form for the events in their lottery machine. They expressed this symmetry in four different ways:

a. symmetrical individual placement of events,
b. symmetrical group placement of events,
c. patterned spatial arrangement and
d. circular spatial arrangement

Constructing sample spaces with symmetrical individual or group placement was the twenty-two out of twenty-three children's attempt to show that they wanted to have two separate sample spaces, one for each colour, with the same structure and working in parallel to collect the same number of points. In one notable example (section 7.3.1.2, figure 7.6), Karen produced a fair sample space by placing groups symmetrically and constructing also a 'brick wall' between them.

A patterned spatial arrangement of the events was another strategy that twelve out of twenty-three children developed for creating a random fair environment. The logic behind this was for balls of one colour to be so near to balls of the other colour that if the white ball touched one colour it would touch the other as well. This construction of fairness was a way to have an equal number of balls without explicit counting (see, for example, Fiona's case in section 7.3.1.3). As already noted, children used patterns very coherently to express the construction of fairness in a random environment. Hence, there was a distinct change in the way they expressed randomness in the sample space - from looking for patterns in random behaviour to building symmetrical patterned sample spaces which have fairness in their outcomes.

A final symmetrical construction that the eight of the twenty-three children developed was making circles. A circle was made to trap the white ball in order for it to touch the balls in the circle the same number of times (e.g. Anne's case in section 7.3.1.4).

Symmetry of placement was a strategy developed 'naturally' by the children (with an exception of one child who constructed an asymmetrical strategy for fairness) while constructing a fair environment, and twenty-two children constructed more than one symmetrical strategy. They judged their construction by using the outcomes of the game, that is, using the aggregate view of their results, it was notable that most of their constructions were successful.

Fifteen of the twenty-three children expressed fairness by constructing *asymmetrical* spatial representations. There were two categories of this: a. equal number and size of two events in a spatial arrangement, and b. 'mixing up' of events.

The first asymmetrical strategy (equal number and size of two events) arose when the children wanted to have two balls in their lottery machine and they were not concerned about their arrangement. Fairness in this case was achieved by the placing of the balls, controlling the size of the balls and making connections between the spatial appearance of sample space and the possible outcome of the game in the longer term (see, for example, Jane's and Mathew's cases in section 7.3.2.1).

'Mixing up' events in the lottery machine was a construction for fairness which arose after the children had watched the continuous movement of the white ball and realised that it did not follow a patterned movement. This strategy was expressed by half of the children. The mixed up balls involved either equal number or/and size of balls (see, for example, Lucy's case in section 7.3.2.2.1) or different number and/or size of balls (e.g. Tom's case in section 7.3.2.2.2). The goal of both strategies was for the two events to have equal probability distributions and hence, the mixing up strategy can be seen as a situated abstraction of an equal probability distribution expressed of having events with different probability to occur.

Symmetric fairness was built as a result of what the twenty-two out of twenty-three children already had in mind of what is fair – that there is a connection between symmetry with fairness, and thus they built a symmetrical environment to bring fairness into their game. From the historical discussion (cf. 1.3.2 and 1.3.4) it is clear that the idea of symmetry co-evolved with ideas of probability. It seems that the spatial metaphor allowed the children of the study to express ideas through symmetry for probabilistic constructions of fairness in line with historical evolution. The idea of asymmetric fairness seems to have developed as the children interacted with the game, checking by the outcome whether their strategy did or did not generate a fair result. So, there appears to be a distinction between knowing in advance what brings fairness in the game (i.e. symmetry), and developing an asymmetric arrangement by checking repeatedly if it brings a fair result. Pratt (1998) argues that a typical characteristic of the way people think about fairness is that they think about equiprobable outcomes and they relate probability to symmetry. Borovcnik and Bentz (1991) also describe that symmetry played a key role in the history of probability, and they conclude that the concept of sample space with equally probable events cannot be fully understood if it is not related to intuitions of symmetry.

The findings of the current study showed that symmetry is, for the twenty-two out of the twenty-three children of the study, also a characteristic of fairness and randomness. The findings agree with what Pratt and Noss (1998) have also shown that existing intuitions about symmetry and fairness derived from interacting with their microworld. The framework of Jones et al (1997; 1999) implies that the children in this study should be categorised at the first level (they should not be able to predict fair and unfair probability situations), however the children were not only able to distinguish fair from unfair probability situations, but were also able to construct fairness in a random environment. Moreover, this study has shown that all the children *intuitively* wanted to construct a fair random environment, by seeing the game as consisting of two "teams", the blue and the red. In the final iteration the children had the opportunity to express both symmetric and asymmetric fairness, as the specified task was simply to construct a fair spatial random environment. Piaget and Inhelder (1975) described symmetry to be an obstacle for children of this age to appreciate randomness. However, this study has shown that children's intuitions of symmetry can be used to construct the idea of fairness in a random environment and even that children could express fairness by using asymmetry to express distributional ideas.

As the children built their symmetric and asymmetric representations of fairness, they had to engage with the ideas of equiprobable sample space and even distribution. It appears that, implicitly, the fifteen out of the twenty-three children were thinking about distribution in ways that are surprisingly related with asymmetrical constructions. Expressing the idea of fairness by the even distribution is an example of a situated abstraction of asymmetric placement, as if the children were thinking of placing balls asymmetrically in a way to 'balance' the sample space. Thus, children constructed symmetric fairness to express a situated abstraction of equiprobable sample space.

9.4.4 Unfairness: Inventing the idea of distribution

The children expressed the situated abstraction that by unbalancing the sample space the distribution is skewed.

Thus, it can be claimed that in the children's construction of unfairness there is a situated abstraction of the distributional idea, in that when they unbalanced the sample space their intention was to unbalance the outcome as well. The construction of fairness required children to deal with unfair situations as well, because their fair construction was not working. The children of the study also faced a problematic situation of unfairness in which they intentionally wanted one of the two planets and one of the teams to get more points than the other. The spatial environment of the game encouraged some children to base their construction of unfairness not only on changing the probability of an event by increasing the quantity of events, but also by increasing the likelihood of an event, by changing the distribution. The study showed that children of the study employed the following representational forms for constructing an unfair random environment:

a. two events with different numbers of same size balls. In this category, twenty-one out of twenty-three children seemed to understand that having more balls of one colour in their sample space makes the game unfair (see John's case in section 7.4.1)

b. events with different sizes of balls but the same number. The different space that each ball occupied in the sample space was a criterion for thirteen out of twenty-three children to recognise an unfair environment (e.g. Anne's case in section 7.4.2)

c. spatial arrangements for unfairness. Here, ten out of twenty-three children expressed the idea of having the same number and size of balls, but they arranged them in a way to make them feel like 'not cheating'. This category suggests an expression of distribution (e.g. Anne's case in section 7.4.3)

The children of the study also constructed unfair environments when they wanted to construct impossible or certain events - when they wanted to make the space kid touch definitely one of the two planets. It was very obvious for them that by not having any balls of a particular colour in the sample space it would be impossible for this colour to have an outcome and it would be certain for the other colour to win (e.g. Karen's case in section 7.4.4). Twenty-one of the twenty-three children of the study expressed a sign of uncertainty of an event to happen for sure when two events existed in their lottery machine (e.g. Demis' case in section 7.4.4). They realised that if an event exists in their distribution, it has a possibility to occur, and that extreme variability is also a possibility. The findings here suggest a way in which children's thinking about distribution developed. At first there was a connection between distribution and randomness. The children's interaction with the game seemed to help them to link the concept of distribution with the existence of two events, possibility of an event and the existence of only one event in their distribution. They also seemed to connect distribution with the size, the amount and the place of each event (the coloured balls) in their lottery machine.

The findings of the study suggest that these young children had a clear idea of what are certain, possible and impossible events. The findings diverge from those of Fischbein, Nello and Marino's (1991) (cf. section 2.4.1), that elementary students do not have in mind a clear definition of such events. They claim that elementary students had difficulty in understanding the term 'certain', when it is related to a compound event and some subjects confused rare with impossible. But, my argument in this thesis is that it is quite possible for children not to be able to *define* what is certain or impossible, but that it is possible for them to *express* these notions in an appropriately designed computational environment.

The children of this study implicitly used the idea of distribution in the sense of changing the likelihood of an event. The game encouraged seventeen of the twenty-three children to base their construction of unfairness, and changing the probability of an event, not only by changing the quantity of that event in the sample space, but also by changing the distribution. The study investigated a different perspective to that of Piaget and Inhelder (1975)'s study, who propose that at this age there is an absence of an appreciation for distribution of the whole. Piaget and Inhelder claim that the problem of random mixture brings up the problem of the forms of distributions and that the final positions of the elements in the mix necessarily take on certain distributive forms of the whole. In this study, the children articulated a need to express distribution in their constructions of the lottery machine and they had a tool to do so. This 'invention' of distribution also emerged in the children's use of asymmetric fairness, as described in the previous section.

9.4.5 Qualitative judgements

In order to make judgements about relative likelihood of an event the children coordinated local and global events. This enabled them to make more qualitative judgements instead of quantitative ones. The children's use of 'proportionality' to control outcomes was restricted to the ratio two to one. However, in a few cases children expressed a belief that this strategy could be generalised.

The children's strategy for recognising what is equal in the game was expressed mainly by looking at the aggregate outcomes of their construction (how close the scorers were, or how near the yellow line the space kid was). The children made judgements about equally likely events based on the global outcomes, even when the 'raw' difference in the outcomes was not quite zero. Without the continuous movement in the computer game, linking sample space and distribution to the outcome, it would have been very difficult for children to do this, because otherwise the outcome would only exist in local terms and the children would not see any global effect directly. Because of the design of the game, the children were able to change focus between local and global events, between quantity and quality: and finally their judgement of their construction to be fair came to be a qualitative one. The findings of this study provide further evidence about the importance of having an experimental task to link local understandings with global ones – see for example the research of Konold and Pollatsek (2002). Having the opportunity to look at the global view of their sample space, the children of this study seemed to be able better to judge their construction (e.g. Anne's case in section 6.4) and to make a step forward from judging their result by focusing on the individual outcome. It was significant that the children started by not distinguishing between the importance of short-term and long-term trials for making judgments, but eventually they could make use of the distinction to construct for equiprobable events.

The children in this study expressed logical comparisons between events to predict whether an event will occur or not. In constructing their lottery machine, children expressed the idea of 'proportional' thinking in three different ways:

a. equality of an event, where twenty-two of the children expressed their understanding that equal ratio would produce an equal outcome (e.g. Victoria's case in section 8.4.1)
b. 'duplicating' points. Ten out of the twenty-three children developed different mechanisms to achieve the goal of one event getting double the points of the other. They tried to manage with a proportion that evidences the concept of twice as much and used the aggregate view of the environment to judge whether it worked or not. However, they did not manage to work effectively with other proportional numbers like tens (e.g. Demis' case in section 8.4.2). This finding comes to agree with Hart's (1984) work in which is argued that doubling is not a ratio.
c. probability of an event. Twenty-two of the children expressed the probability of an event by indicating which colour would get more points than the other. They did not pay attention to how many points the team would get, but they seem to be generally satisfied to find out which would be the winning team. The medium gave to three children an impetus to think that it might be a 'strategy' for guessing the score, but they did not achieve the use of proportional thinking for the probability of an event to occur (e.g. George's case in section 8.4.3).

The findings of the study suggest that children preferred to make logical comparisons instead of arithmetical ones. This partly contradicts Piaget and Inhelder's (1975) work, which finds an absence of any ability to make quantitative comparisons between two sets, since (for them) the child at this level has neither 'elementary logical operations' nor 'arithmetical operations formative of the series of the whole numbers'.

The children of the study focused their attention on qualitative judgements as a way to deal with the relative likelihood of an event. The findings show that the children's 'proportional strategies' were restricted only to doubling, although some children indicated that they could generalise this. For example, George, in section 8.4.2 indicates that because he had two reds and one blue, he might get ten points for the reds and five points of the blues, as two times five makes ten. According to Singer and Resnick's (1992) analysis (cf. section 2.4.1), George's proportional thinking involves a part-part schema - describing the relationship of two parts to each other and not seeing the proportion as a part-whole (i.e. 'two reds out of three and one blue out of three'). In general, the children expressed the part-part schema of proportional thinking than the part-whole schema (see section 2.4.1). They seemed to want to avoid proportional thinking and preferred to start the game and wait long enough in order to see what would happen to the outcomes.

This finding brings us to the next section concerning children's strategies in order to deal with the law of large numbers.

9.4.6 Inventing the law of large numbers

The children employed four distinct strategies to express the idea that their construction could only be judged with respect to a large number of trials.

Children employed the idea of increasing the likelihood of each trial, in order to judge better the success of their construction, by

a. increasing the speed of the white ball, in order to hit the other balls more often (e.g. Anne and Paul, section 8.2.1)[28]
b. increasing the number of events in the lottery machine, by adding more coloured balls (e.g. Simon, section 8.2.2)[29]
c. adding more white balls, making more 'random generators' to produce events (e.g. Fiona, section 8.2.3)[30]
d. making the size of the white ball bigger in order to hit the other balls more often (e.g. Matthew, section 8.2.4)[31]
e. leaving the game running for longer. This was a common strategy; all the twenty-three children expressed it. Helen (section 8.2.4) described her successful construction by saying that it would keep going 'the whole night'

[28] This idea was expressed by 7/23 children, see section 8.3.
[29] This idea was expressed by 16/23 children, see section 8.3.
[30] This idea was expressed by 21/23 children, see section 8.3.
[31] This idea was expressed by 5/23 children, see section 8.3.

and Orestis (section 8.2.4) admitted that his construction did not get the desired result 'yet'.

It is apparent that the game provided children the opportunity to express the idea that stability can come from increasing outcomes in ways different from the 'leaving longer' strategy. The game provided ways in which twenty-one children could engage with the idea of getting 'bigger numbers' besides leaving the game work for longer. It can be said that the above children's expressions is evidence of several situated abstractions for the law of large numbers.

In their constructions, children seemed to express a belief that the mathematical idea of fairness (equal likelihood for example) could only be tested in the long term. This observation contradicts findings in the literature that describe a child's everyday intuition that the more trials there are; the more unsureness there is about the result. For example, Metz (1998) reports that the 'law of small numbers' was the only interpretation underlying the students' acknowledgment of determinism and this interpretation consisted of the belief that the contents of an unknown sample space could be directly assessed through a small number of observations. In this study, children seemed to develop intuitions about the stability of long-term trials, something that the game had afforded by making visible the aggregate outcome of their construction. It can be concluded that the computational medium did support children's intuition of the law of large numbers, giving support to Biehler's (1991) argument that without computer support, it is difficult to work with large numbers, and that long-run frequencies remain mysterious and students will act as if a law of small numbers applies.

9. 5 Some Didactical Implications

This study has suggested that a range of important probabilistic intuitions exist from an early age and that stochastic experiences can be fundamentally important for children's development of intuitions about chance and probability. Young students bring intuitions to the classroom environment, but most of these are neglected because of the absence of teaching instruction. The findings of the study suggest that teaching probability can be based on knowledge of students' intuitions, and an active learning environment can give students the opportunity to construct their own meanings by connecting new information to what they already know. In this study, I have described the evolutions in students' meanings for mathematical knowledge, which occurred while they built computational models of sample space and distribution. This finding provides support to the general constructionist thesis that engagement in the building of some external, shareable and personally meaningful product is conducive to mathematics learning (Papert, 1991).

In the constructionist prespective, the learner becomes involved in creating, as well as appropriating, artefacts that become part of the 'culture' of a learning system. Looking forward, I believe that a well-structured curriculum on probability could be introduced from the early levels of the elementary school. The Cypriot National Curriculum, as well as the National Curriculum of England and Wales, contains almost no work on chance and probability at this level. I suggest that probability should be introduced at this level for a variety of reasons: a. The findings of this study suggest that intuitions on probability exist from an early age and therefore should be exploited before being abused, b. Probability at this age can be connected with games and this can bring positive attitudes to pupils about mathematics and the connection to everyday life, c. It helps one to understand and evaluate information in the world, d. In later years it is a prerequisite to enter many fields of study, and weak foundations of understanding are highly damaging. For these reasons, I believe that the subject of probability should not be neglected but should have a significant place in the elementary curriculum.

9. 6 Limitations of this study

The limitations of this study are those of time, scale and sensitivity of findings to the specific features of the computer game. The time spent with each child was sometimes limited, depending on factors such as the maximum time that the child's teacher permitted. Typically each child engaged in the final computer game individually for between two to three hours for two or three sessions. Longer periods of time may have allowed other meanings to emerge and it is possible that some observations would have proved to be more or less significant than the interpretation given in this account. For this reason, the research can be generalised only with some care. Experience of working with a computer-designed game in this study emphasises the sensitive nature of the relationship between the children's activity and the construction of probabilistic meanings and the tools and structures made available within the microworld. The iterative design methodology helps to gain a feeling for the nature of this relationship, but it is necessary to interpret the findings with some caution.

9. 7 Implications for future research

The findings of this study support the view that there do exist many unstructured pieces of knowledge in young children's thinking about randomness and probabilistic ideas. These pieces of knowledge surfaced while using a medium specially designed for expressing randomness and chance, and I believe they can most easily be identified with difficulty if static, not dynamic, media were involved. If these findings are valid, then we might expect that similar findings would result from research using similarly specially designed tools. The children of the study were taken from a particular age range. One of the reasons for choosing this age range (as described in Chapter Four) was to see whether children's ideas of randomness could be predicted using standard misconceptions as Tversky and Kahneman (1983) describe, or stages of thinking as Piaget and Inhelder (1975), and Jones et al (1997; 1999) describe. The findings suggest that children of this age express themselves in a different way to what is expected from 'stages of thinking' analysis. Further research could be undertaken by using the same or similar tools with children of other ages.

The lottery game offered children the opportunity to manipulate only two different colours of balls. This could be modified to give children the opportunity to express ideas of conditional probability, by having two lottery machines, or by having more than two events in their sample space. This might reveal other aspects of intuitive probabilistic knowledge. The study has also shown some of the probabilistic geometrical intuitions that children have; these appear to be quite novel in research in learning probability and provide an inspiration for further research. A further study could also be undertaken with children using the Pathways software actually to program games and find out how meaning of probabilistic ideas can be achieved through the expressiveness of programming.

Further research based on the present findings could be also used to identify a broader range of applicability. It would be interesting to see the game performed by children of different cultures in order to see whether socio-cultural variables affect probabilistic intuitions. In this respect, the software could also be used in the sense of social constructivism. The game used in this study could also be used in curriculum-oriented research on probability, leading to a methodology for teaching and learning conceptions of probability in a classroom environment.

BIBLIOGRAPHY

Acredolo, C., O'Connor, J., Banks, L., and Horobin, K. (1989). 'Children's Ability to Make Probability Estimates: Skills Revealed through Application of Anderson's Functional Measurement Methodology' Child Development, 60, p. 933-945.

Ahlgren, A., and Garfield, J. (1991). 'Analysis of the Probability Curriculum' in Kapadia, R. and Borovcnik, M. (eds.) Chance Encounters: Probability in Education Dordrecht: Kluwer, p. 107-134

Amir, G., and Williams, J. (1994). 'The influence of children's culture on their probabilistic thinking' Proceedings of the Eighteenth Annual Conference of the International Group for the Psychology of Mathematics Education Vol. 2 p. 24-31, Lisbon, Portugal

Aspiwall, L. and Tarr, J. (2001). 'Middle school students' understanding of the role sample size plays in experimental probability' The Journal of Mathematical Behavior, 20, 2, p.229-245

Ayres, P. and Way, J. (1999). 'Decision-making strategies in probability experiments: the influence of prediction confirmation' Proceedings of the Twenty Third Annual Conference of the International Group for the Psychology of Mathematics Education Vol.2, p. 41-48, Haifa: Israel Institute of Technology

Ayres, P. and Way, J. (2000). 'Knowing the sample space or not: The effects on decision making' Proceedings of the Twenty Fourth Annual Conference of the International Group for the Psychology of Mathematics Education Vol.2, p. 33-40, Hirosima University, Japan

Balacheff, N. and Kaput, J. (1996). 'Computer-Based Learning Environments in Mathematics' in Bishop, A., Clements, K., Keitel, C., Kilpatrick, J. and Laborde, C. (eds.) International Handbook of Mathematics Education, Kluwer Academic Publishers, p. 469-501

Barnett, M., and Noss, R. (1997). Children's Mathematical Understanding of Probability and Risk (research perspective) London: Institute of Education

Batanero, C., Serrano, I. (1999). 'The Meanings of Randomness for Secondary School Students' Journal of Research in Mathematics Education, 30, 5, p. 558-567

Batanero, C., Serrano, I., and Garfield, J. (1996). 'Heuristics and Biases in Secondary School Students Reasoning about Probability' Proceedings of the Twentieth Annual Conference of the International Group for the Psychology of Mathematics Education Vol. 2, p. 51-57, Valencia, Spain

Ben-Zvi, D. and Arcavi, A. (2001). 'Junior High school students' constructions of global views of data and data representations' Educational Studies in Mathematics, 45, 35-65

Biehler, R. (1991). 'Computers in Probability Education' Probability' in Kapadia , R., and Borovcnik, M. (eds.) Chance Encounters: Probability in Education Dordrecht: Kluwer, p. 169-211

Booth, R., and Thomas, M. (2000). 'Visualization in Mathematics Learning: Arithmetic Problem-solving and Student Difficulties' Journal of Mathematical Behavior, 18, 2, p. 169-190

Borovcnik, M., and Bentz, H. (1991). 'Empirical Research in Understanding Probability' in Kapadia , R., and Borovcnik, M. (eds.) Chance Encounters: Probability in Education Dordrecht: Kluwer, p. 73-105

Borovcnik, M., and Peard, R. (1996). 'Probability' in Bishop, A., Clements, K., Keitel, C. Kilpatrick, J. and Laborde, C. (eds.) International Handbook of Mathematics Education Netherlands: Kluwer Academic Publishers, p. 239-287

Bruner, J. S. (1966). Studies in Cognitive Growth New York: John Wiley and Sons

Bruner, J. S. (1974). Relevance of Education Oxford: Oxford University Press

Burgess, R. (1984). In the Field New York: Routledge

Chiu, M. (1996). 'Exploring the origins, uses, and interactions of student intuitions: comparing the lengths of paths' Journal for Research in Mathematics Education, 27, 4, p. 478-504

Clapham, C. (1990). The Concise Oxford Dictionary of Mathematics (second edition) Oxford: Oxford University Press

Cunningham, S. (1991). 'The visualization Environment for Mathematics Education' in Zimmermann, W. and Cunningham, S. (eds.) Visualization in Teaching and Learning Mathematics Washington, D.C.: Mathematical Association of America, p. 67-76

Cohen, L. (1979). 'On the psychology of prediction: Whose is the fallacy?' Cognition, 7, p. 385-407

Daly, F., Hand, D., Jones, M., Lunn, A., and McConway, K. (1995). Elements of Statistics London: The Open University

David, F. (1962). Games, Gods and Gambling London: Griffin

Davis, R. (1984). Learning Mathematics. The cognitive science approach to Mathematics Education London: Routledge

Denis, M. (1991). Image and Cognition London: Harvester Wheatsheaf

Denscombe, M. (1998). The Good Research Guide for small-scale social research projects Buckingham: Open University Press

Dewey, J. (1902). The Child and the Curriculum. Chicago: University of Chicago Press.

diSessa, A. (1986). 'Artificial Worlds and Real Experience' Instructional Science, 14, p. 207-227

diSessa, A. (1988). 'Knowledge in Pieces' in G. Forman and P. Pufall (eds.) Constructivism in the Computer Age New Jersey: Lawrence Erlbaum Associates, p.49-70 diSessa, A. (1989). 'A Child's Science of Motion: Overview and First Results' in U. Leron and N. Krumholtz (eds.) Proceedings of the Fourth International Conference for Logo and Mathematics Education, Jerusalem

diSessa. A. (1995). 'The Many Faces of a Computational Medium: Teaching the Mathematics of Motion' in diSessa, A., Hoyles, C. and Noss, R. (eds.) Computers and Exploratory Learning Germany: ASI, p. 337-359

diSessa, A. (2000). Changing Minds: Computers, Learning, and Literacy Massachusetts: The MIT Press

diSessa. A., Hammer, D., and Sherin, B. (1991). 'Inventing Graphing: Meta-Representational Expertise in Children' Journal of Mathematical Behavior, 10, 2, p. 117-160

Dreyfus, T. (1991) 'Advanced Mathematical Thinking Processes' in Tall, D. (eds). Advanced Mathematical Thinking London: Kluwer Academic Publishers, p. 25-41

Dreyfus, T. (1993). 'Didactic Design of Computer-based Learning Environments' in Keitel, C. and Ruthven, K. (eds.) Learning from Computers: Mathematics Education and Technology, NATO ASI Series, Vol. F 121, Springer-Verlag, Berlin, 1993, p. 101-130

Edwards, L. D. (1998). ' Embodiments of mathematics and science: Microworlds as representations. Journal of Mathematical Behavior, 17, 1, p. 53-78

Falk, R. (1992). 'A closer look at probabilities of the notorious three prisoners' Cognition, 43, p. 197-223

Falk, R., Falk, R., and Levin, I. (1980). 'A potential for learning probability in young children' Educational Studies in Mathematics, 11, p. 181-204

Falk, R. and Konold, C. (1994). 'Random Means Hard to Digest' Focus on Learning Problems in Mathematics, 16, 1, p. 2-11

Feller, W. (1968). An introduction to Probability Theory and Its Applications Vol. 1 (Third Edition New York: John Willey and Sons, Inc.

Ferguson, G. (1971). Statistical analysis in psychology and education (3rd ed.) New York: MacGraw-Hill

Fischbein. E. (1975). The intuitive sources of probabilistic thinking in children London: Reidel

Fischbein, E. (1982). 'Intuition and Proof' For the Learning of Mathematics, 3, 2, p. 9-19

Fischbein, E. (1987). Intuition in Science and Mathematics Dordrecht: Reidel

Fischbein, E. (1999a). 'Intuitions and schemata in mathematical reasoning' Educational Studies in Mathematics, 38, p. 11-50

Fischbein, E. (1999b). 'Psychology and mathematics education' Mathematical Thinking and Learning, 1, p. 47-58

Fischbein, E., Nello, M., and Marino, M. (1991). 'Factors affecting probabilistic judgements in children and adolescents' Educational Studies in Mathematics, 22, p. 523-549

Fischbein, E.. and Schnarch, D. (1996). 'Intuitions and Schemata in Probabilistic Thinking' Proceedings of the Twentieth Annual Conference of the International Group for the Psychology of Mathematics Education Vol. 2, p. 353-360, Valencia, Spain

Fischbein, E.. and Schnarch, D. (1997). 'The Evolution With Age of Probabilistic, Intuitively Based Misconceptions' Journal for Research in Mathematics Education, 28, 1, p. 96-105

Garfield, J., and Ahlegren, A. (1988). 'Difficulties in Learning Basic Concepts in Probability and Statistics: Implications for Research' Journal for Research in Mathematics Education, 19, 1, p. 44-63

Gray, E. and Tall, D. (2002). 'Abstraction as a natural process of mental compression' Proceedings of the Twenty Sixth Annual Conference of the International Group for the Psychology of Mathematics Education Vol.1, p. 115-119, Norwich, United Kingdom

Green, D. (1983). 'A Survey of Probability Concepts in 3000 Pupils Aged 11-16 Years' Proceedings of the First International Conference on Teaching Statistics Vol. 2 Teaching Statistics Trust, p. 766-783

Greer, B. (2001). 'Understanding probabilistic thinking: The legacy of Efraim Fischbein' Educational Studies in Mathematics, 45, p. 15-33

Grimment, G. and Stirzaker, D. (1992). Probability and Random Processes (2nd ed.) Oxford: Clarendon Press

Hacking, I. (1975). The emergence of probability Cambridge: Cambridge University Press

Hadamard, J. (1945). The Psychology of Invention in the Mathematical Field Princeton: Princeton University Press

Harel, I. (1991). Children designers: interdisciplinary constructions for learning and knowing mathematics in a computer-rich school New Jersey: Ablex

Harel, I. and Papert, S. (1990). 'Software Design as a Learning Environment' in Harel, I. (ed.) Constructionist Learning Massachusetts: MIT Media Laboratory, p. 19-50

Hart, K. M. (1984). Ratio: Children's Strategies and Errors. Windsor: NFER-Nelson.

Hawkins, A., and Kapadia, R. (1984). 'Children's conceptions of probability- a psychological and pedagogical review' Educational Studies in Mathematics, 15, p. 349-377

Hawkins, A., and Hawkins, P. (1997). 'Are lawyers prey to probability misconceptions irrespective of mathematical education?' Proceedings of the Twenty First Annual Conference of the International Group for the Psychology of Mathematics Education Vol. 3 p.41-48.

Heyman, B. (1998). Risk, health and health care London: ARNOLD

Heyman, B., and Henriksen, M. (1998) 'Probability and health risks' Risk, health and health care London: ARNOLD, p. 65-105

Hoemann, H., and Ross, B. (1971). 'Children's understanding of probability concepts' Child Development, 42, p. 221-236

Hoyles, C. (1993). 'Microworlds/School worlds: The transformations of an innovation' in Keitel, C. and Ruthven, K. (eds.) Learning Through Computers: Mathematics and Educational Technology, Springer Verlag, Berlin, p. 1-17

Hoyles, C. and Noss, R. (eds.) (1992). 'Learning Mathematics and Logo' London: The MIT Press

Jacobs, J., and Potenza, M. (1991). ' The Use of Judgement Heuristics to Make Social and Object Decisions: A Developmental Perspective' Child Development, 62, p. 166-178

Jones, G., Langrall, C., Thornton, C., and Mogill, A. (1997). 'A framework for assessing and nurturing young children's thinking in probability' Educational Studies in Mathematics, 32, p. 101-125

Jones, G., Langrall, C., Thornton, C., and Mogill, A. (1999). 'Using students' probabilistic thinking in instruction' Journal of Research in Mathematics Education, 30, p. 487-519

Jones, G., Langrall, C., Thornton, C., and Nisbet, S. (2002). 'Elementary Students' Access to Powerful Mathematical Ideas' in English, L. (ed.) Handbook of International Research in Mathematics Education New Jersey: LEA Publishers, p. 113-141

Kafai, Y., Franke, M., Ching, C. and Shih, J. (1998). 'Game design as an interactive learning environment for fostering students' and teachers' mathematical inquiry' International Journal of Computers for Mathematical Learning, 3, p. 149-184

Kafai, Y., and Resnick, M. (1996). Constructionism in Practice New Jersey: LEA

Kahneman, D., and Tversky, A. (1973). 'On the Psychology of Prediction' Psychological Review, 80, 4, p. 237- 251

Kahneman, D., and Tversky, A. (1982). 'On the study of statistical intuitions' Cognition, 11, p. 123-141

Kalas, I., Blaho, A. (2002) 'Exploring visible mathematics with IMAGINE: Building new mathematical cultures with a powerful computational system. Learning in School, Home and Community, p. 53-64

Kant, I. (1980). Critique of Pure Reason (translated by N.K. Smith) London: The Macmillan Press Ltd

Kapadia, R. and Borovcnik, M. (1991). 'The Educational Perspective' in Kapadia, R. and Borovcnik, M. (eds.) Chance Encounters: Probability in Education Dordrecht: Kluwer Academic Publishers, p.1-26

Kaput, J. (1995). 'Creating Cybernetic and Psychological Ramps from the Concrete to the Abstract: Examples from Multiplicative Structures' in Perkins, D., Schwartz, J., West Marwell, M., Wiske Stone, M. (eds.) Software Goes to School: Teaching for Understanding with New Technologies, p. 130-154

Kaput, J., Noss, R. and Hoyles, C. (2002). 'Developing New Notations for a Learnable Mathematics in the Computational Era' in English, L. (ed.) Handbook of International Research in Mathematics Education New Jersey: LEA Publishers, p. 51-75

Konold, C. (1989). 'Informal Conceptions of Probability' Cognition and Instruction, 6, 1, p. 59-98

Konold, C. (1991). 'Understanding students' beliefs about probability' in von Glasersfeld, E. (ed.) Radical Constructivism in Mathematics Education Holland: Kluwer, p. 139-156

Konold, C. and Pollatsek, A. (2002). 'Data Analysis as the Search for Signals in Noisy Porcesses' Journal for Research in Mathematics Education, 33, 4, 259-289

Konold, C., Pollatsek, A., Well, A., Lohmeier, J., and Lipson, A. (1993). 'Inconsistencies in students' reasoning about probability' Journal for Research in Mathematics Education, 24, 5, p. 392-414

Kuzmak, S. and Gelman, R. (1986). 'Young Children's Understanding of Random Phenomena' Child Development, 57, p. 559-566

Lajoie, S., Jacobs, V., and Lavigne, N. (1995). 'Empowering Children in Use of Statistics' Journal of Mathematical Behavior, 14, p. 401-425

Larkin, J. and Simon, H. (1987). 'Why a diagram is (sometimes) worth ten thousand words' Cognitive Science, 11, p. 65-99

Li, J. and Pereira-Mendoza, L. (2002) 'Misconceptions in probability' training' Proceedings of the 6th International Conference of Teaching Statistics (ICOTS 6) Cape Town

Lopes, C. and de Moura, A. (2002) 'Probability and statistics in elementary school: A research of teachers' training' Proceedings of the 6th International Conference of Teaching Statistics (ICOTS 6) Cape Town

Marschall, S. (1990). The Assessment of Schema knowledge for Arithmetic Story Problems: A cognitive science prospective' in Kulu, G. (ed.) Assessing Higher Order Thinking in Mathematics American Association for the Advancement of Science, p. 155-168

Metz, K. (1998). 'Emergent Understanding and Attribution of Randomness: Comparative Analysis of Reasoning of Primary Grade Children and Undergraduates' Cognition and Instruction, 16, 3, p. 285-365

Moore, D. (1990). 'Uncertainty' in Steen, L. A. (ed.) On the shoulders of giants: New approaches to numeracy Washington: National Academy Press, p. 95-137

Nisbett, R., Krantz, D. Jepson, C. and Kunda, Z. (1983). 'The Use of Statistical Heuristics in Everyday Inductive Reasoning' Psychology Review, 90, 4, p. 339-363

Noss, R. (2001). 'For a Learnable Mathematics in the Digital Culture' Educational Studies in Mathematics, 48, p. 21-46

Noss, R., Healy, L. and Hoyles, C. (1997). 'The construction of mathematical meanings: connecting the visual with the symbolic' Educational Studies in Mathematics, 33, p. 203-233

Noss, R., and Hoyles, C. (1996). Windows on Mathematical Meanings: Learning Cultures and Computers. Dordrecht: Kluwer

Noss, R., Hoyles, C. and Pozzi, S. (2002). 'Abstraction in Expertise: A Study of Nurses' Conceptions of Concentration' Journal for Research in Mathematics Education, 33, 3, p. 204-229

Nunes, T., Schliemann, D., and Carraher, D. (1993). Street Mathematics and School Mathematics USA: Cambridge University Press

Ojeda, A. (1999). 'The research of ideas of probability in the elementary level of education' Proceedings of the Twenty Third Annual Conference of the International Group for the Psychology of Mathematics Education , Vol. 4, p. 1-8, Haifa: Israel Institute of Technology

Paparistodemou, E. and Philipou, G. (2002). 'Intuitions of the Concept of Probability in 6-7 year-old Children' Themes in Education, 3, 1, p. 63-78

Papert, S. (1980). Mindstorms New York: Harvester Wheatsheaf

Papert, S. (1990). 'Introduction' in Harel, I. Constructionist Learning Massachusetts: MIT Media Laboratory, p. 1-8

Papert, S. (1991). 'Situating Constructionism' in Harel, I. and Papert, S. Constructionism New Jersey: ABLEX, p. 1-11

Papert, S. (1996). 'An exploration in the space of mathematics educations' International Journal of Computers for Mathematical Learning, 1, p. 95-123

Petocz, P. and Reid, A. (2002). 'How students experience learning statistics and teaching' Proceedings of the 6th International Conference of Teaching Statistics (ICOTS 6) Cape Town

Philips, D. C. (1995). 'The Good, the Bad, and the Ugly: The Many Faces of Constructivism' Educational Researcher, 24, p. 5-12

Piaget, J. (1929). The Child's Conception of the World New York: Harcourt Brace

Piaget, J. (1952). The Origins of Intelligence in Children. New York: International University Press

Piaget, J. (1974). Understanding Causality New York: W.W. Norton and Company

Piaget, J. and Inhelder, B. (1975). The origin of the idea of chance in children London: Routledge and Kegan Paul

Piaget, J. and Szeminska, A. (1952). The child's conception of number London: Routledge and Kegan

Powney, J. and Watts, M. (1987). Interviewing in educational research London: Routledge

Pratt, D. and Noss, R. (1996). ' Designing a Domain for Stochastic Abstraction' Proceedings of the Twentieth Annual Conference of the International Group for the Psychology of Mathematics Education Vol. 4, p. 163-170, Valencia, Spain

Pratt, D. (1998). The Construction of Meanings *In* and *For* a Stochastic Domain of Abstraction (doctoral dissertation) London: Institute of Education, University of London

Pratt, D. and Noss, R. (1998). 'The Co-ordination of Meanings for Randomness' Proceedings of the Twenty Second Annual Conference of the International Group for the Psychology of Mathematics Education Stellenbosch, South Africa

Pratt, D. (2000). 'Making sense of the Total of Two Dice' Journal for Research in Mathematics Education, 31, 5, p. 602-625

Presmeg, N. (1986). 'Visualisation in High School Mathematics' For the Learning of Mathematics, 6, 3, p. 42-47

Robson, C. (1993). Real World Research Oxford: Blackwell

Ross, S. (2002). A First Course in Probability (6th edition) New Jersey: Prentice Hall

Rubin, A. (2002). 'Interactive Visualizations of Statistical Relationships: What do we gain?' Proceedings of the 6th International Conference of Teaching Statistics (ICOTS 6) Cape Town

Schoenfeld, A. (1985). 'Metacognitive and Epistemological Issues in Mathematical Understanding' in Silver, E. (ed.) Teaching and Learning Mathematical Problem Solving New Jersey: LEA, p. 361-379

Scott, D. and Usher, R. (1999). Researching Education London: Cassell

Shaughnessy, J. (1985). ' Problem-Solving Derailers: The Influence of Misconceptions on Problem-Solving Performance' in Silver, E. (ed.) Teaching and Learning Mathematical Problem Solving New Jersey: LEA, p. 399-415

Shaughnessy, J. (1992). 'Research in probability and statistics: reflections and directions' in Grouws, D. (ed.) Handbook of Research on Mathematics Teaching and Learning (NCTM) New York: Macmillan Publishing Company, p. 465-495

Sierpinska, A, (2002). 'Reaction' Proceedings of the Twenty Sixth Annual Conference of the International Group for the Psychology of Mathematics Education Vol.1, p. 129-133, Norwich, United Kingdom

Singer, J. and Resnick, L. (1992). 'Representations of proportional relationships: Are children part-part or part-whole reasoners?' Educational Studies in Mathematics, 23, p. 231-246

Smith, J., diSessa, A., and Rochelle, J. (1993). 'Misconceptions Reconceived: A Constructivist Analysis of Knowledge in Transition' The Journal of the Learning Sciences, 3, 2, p. 115-163

Solomon, C. (1986). Computer Environments for Children Cambridge: The MIT Press

Steinbring, H. (1991). 'The Theoretical Nature of Probability in the Classroom' in Kapadia, R. and Borovcnik, M. (eds.) Chance Encounters: Probability in Education Dordrecht: Kluwer, p. 135-167

Stohl, H. and Tarr, J. (2002). 'Developing notions of inference using probability simulations tools' The Journal of Mathematical Behavior, 21, 3, p. 319-337

Sutherland, R. and Balacheff, N. (1999). 'Didactical complexity of computational environments for the learning of mathematics' International Journal of Computers for Mathematical Learning, 4, p. 1-26

Szendrei, J. (1996). 'Concrete Materials in the Classroom' in Bishop A. et al. (eds.) International Handbook of Mathematics Education, p. 411-434

Tall, D. (1991). 'The psychology of advanced mathematical thinking' in Tall, D. (eds). Advanced Mathematical Thinking London: Kluwer Academic Publishers, p. 3-21

Tall, D. and Vinner, S. (1981). 'Concept image and concept definition in mathematics with particular reference to limits and continuity' Educational Studies in Mathematics, 12, p. 151-169

Truran,K. and Truran, J. (1999). 'Are dice independent? Some responses from children and adults' Proceedings of the Twenty Third Annual Conference of the International Group for the Psychology of Mathematics Education Vol.4, p. 289-296 Haifa: Israel Institute of Technology

Turkle, S. and Papert, S. (1991). 'Epistemological Pluralism and the Revaluation of the Concrete' in Harel, I. and Papert, S. (eds.) Constructionism New Jersey: Ablex, p. 161-192

Tvesrky, A. and Kahneman, D. (1983). 'Extensional Versus Intuitive Reasoning: The Conjunction Fallacy in Probability Judgement' Psychological Review, 90, 4, p. 293-315

Vinner, S. (1983). 'Concept definition, concept image and the notion of function' International Journal of Mathematics Education and Science Technology ,14, 3, p.293-305

Vygotsky, L.S. (1978). Mind in society. Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1981). 'The instrumental method in psychology' in Wertsch, J. V. (ed.) The concept of activity in Soviet psychology Armonk, NY: M.E. Sharpe, p. 134-143

Watson, J. and Moritz, J. (2000). 'Developing Concepts of Sampling' Journal for Research in Mathematics Education, 31, 1, p. 44-70

Wilensky, U. (1991). 'Abstract Meditations of the Concrete and Concrete Implications for Mathematics Education' in Harel, I. and Papert, S. Constructionism New Jersey: ABLEX, p. 193-203

Wilensky, U. (1993). Connected Mathematics: Building Concrete Relationships with Mathematical Knowledge (doctoral dissertation) Cambridge: MIT

Wilensky, U. (1995). 'Paradox, Programming, and Learning Probability: A Case Study in a Connected Mathematics Framework' Journal of Mathematical Behavior, 14, p. 253-280

Wilensky, U. (1997). 'What is normal anyway? Therapy for epistemological anxiety' Educational Studies in Mathematics, 33, p. 171-202

Zimmermann, W. and Cunningham, S. (1991). 'Editor's Introduction: What is Mathematical Visualization?' in Zimmermann, W. and Cunningham, S. (eds.) Visualization in Teaching and Learning Mathematics Washington, D.C.: Mathematical Association of America, p. 1-8

APPENDICES

A1 Snapshot of tilt box experiment

R: What do you think will happen when I tilt this box to the other side?
P: These balls will go to the other side.
R: Where will the red balls go?
P: The reds will move here, then the yellows, the greens here and the oranges here.
R: Can you show me how the reds will move?
P: The first red **might** go here...
R: What about the yellow one?
P: The **first** one?
R: Yes.
P: It will move here... Tilt it to have a look!
(The box is tilted.)
P: Oops... They messed up.
R: Why is that?
P: Maybe because the yellow fell in front of the purple and they messed up.

A2 Children's experience with Pathways

A2.1 Showing and Reacting to Colours

- Activity 1: Off the computer

Think of three different colours and draw them in the boxes below:

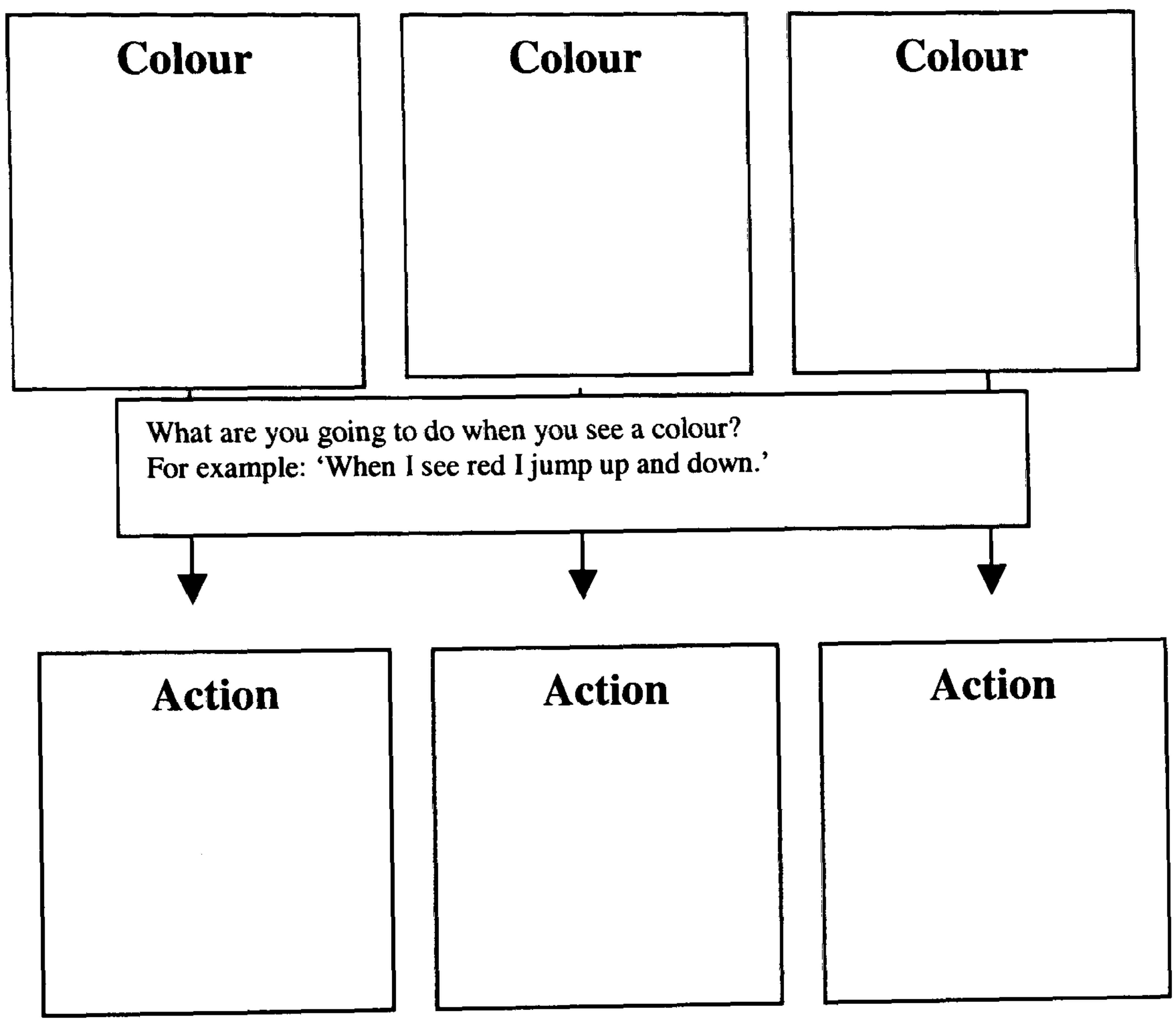


Diagram A1.1: The showing and reacting to colours diagram

Now, I will show you a colour and act according to what you filled in the above diagram.

- Activity 2: On the computer

Play the coloured planets game.

Figure A1.1: The coloured planets game

Try to read the rules of the planets and the scorer.

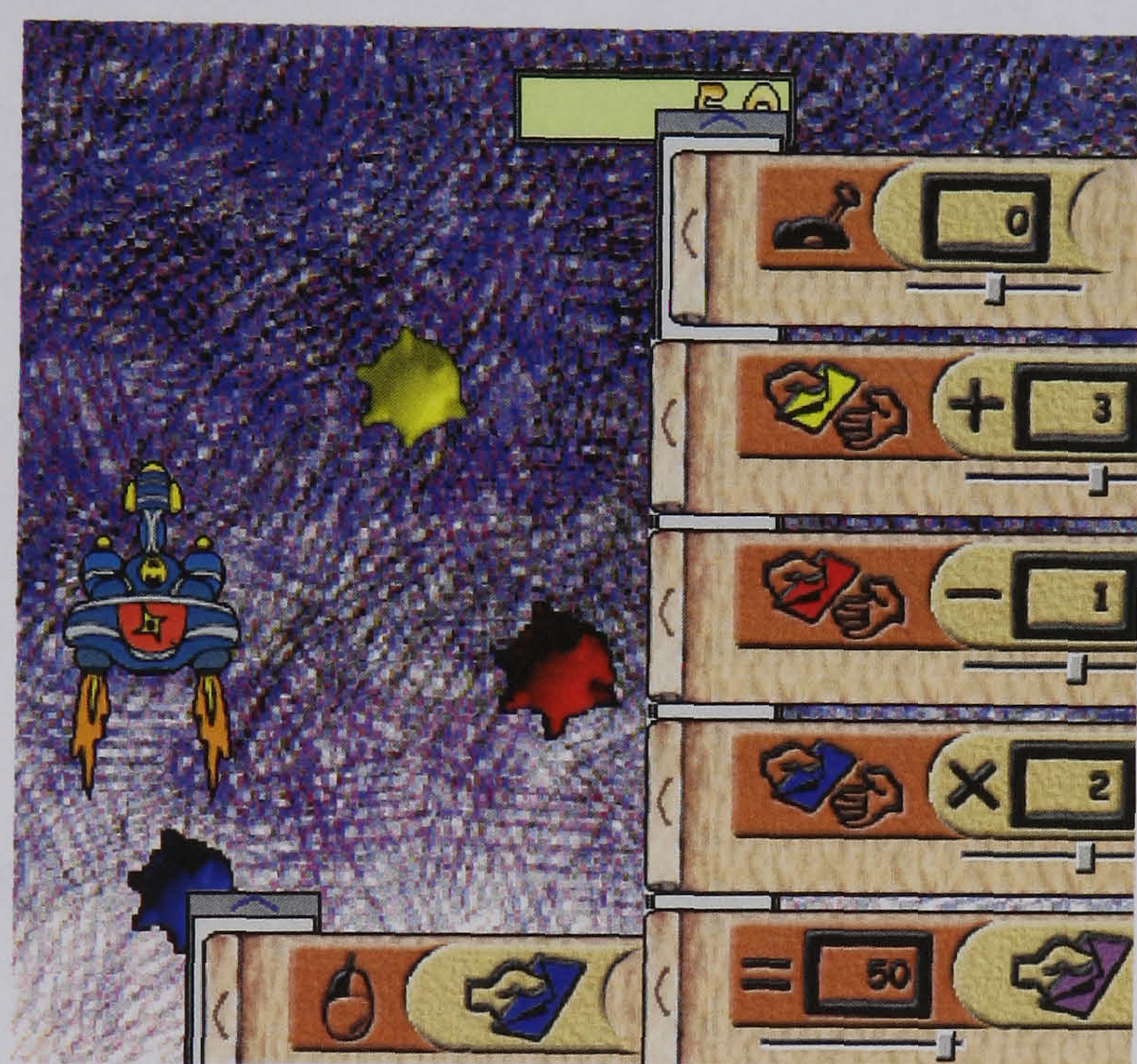


Figure A1.2: The rules of the planets and the scorer

Can you now fill in the table?

Blue planets:

When do they show a colour?	What colour do they show?		Which thing reacts?	What does it do?

Table A1.1: The blue planets

- Activity 3

Play the game again.

Can you change the rules of the game in order to stop when the result of the scorer is 50?

Describe what you have done.

A2.2 Help the duckling

You are going to make a game where a duckling tries to meet the gold fish and collect magic wishes!

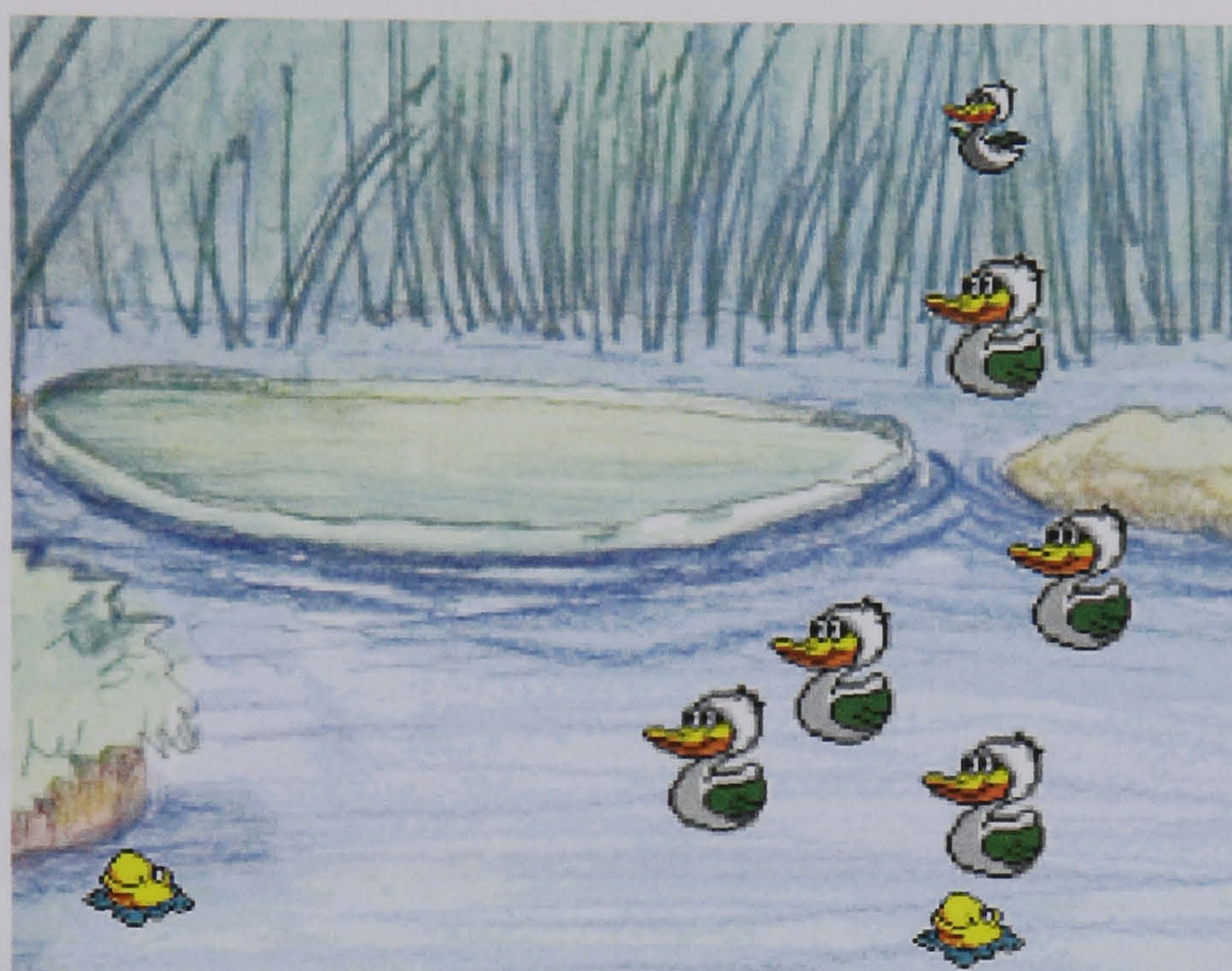


Figure A1.3: The duckling game

- Activity 1: Looking at the duckling!

Find the duckling.

Look at its rules.

Try to read and understand them.

Prediction question:

Do you think that the duckling can meet all the fish?

Now, play the game….

Did the duckling meet the fish?

Why yes or why not?

- Activity 2: Can you help the duckling?

Look again at the rules of the duckling!

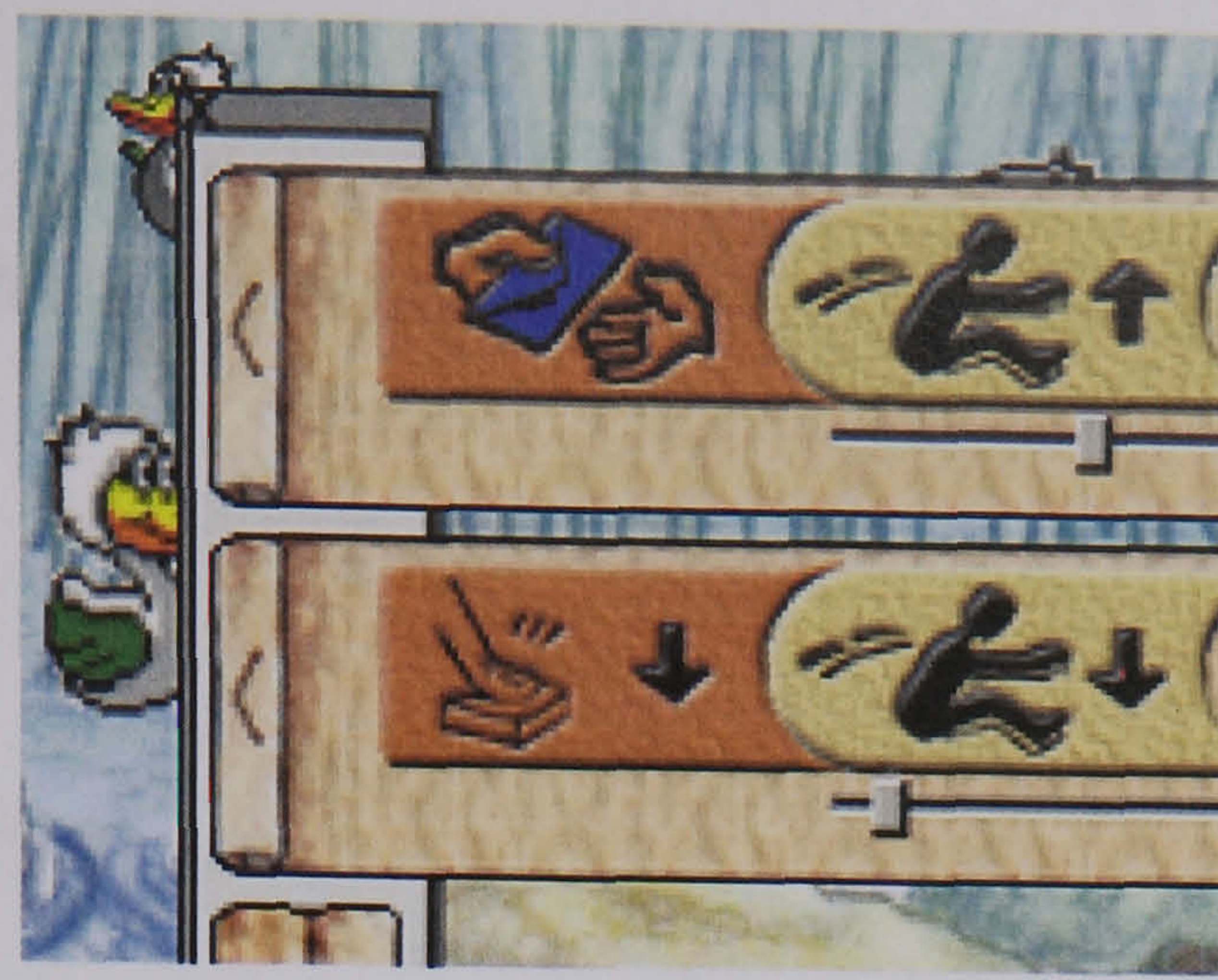


Figure A1.4: The rules of the duckling

Which rule helps the duckling to move towards fish?

See what happens when you change the arrows in the second rule.

Try to add new rules to the duckling in order to reach all the fish. You can use the stones and the toolbox.

What did you do?

Prediction question:

Do you think that the duckling now can reach all the fish?

Play the game again.

Did the duckling reach the fish?

Why yes or why not?

- Activity 4: Another change!

Can you train one of the ducks to make another sound when it touches the duckling?

Try it by using the stone that plays a sound:

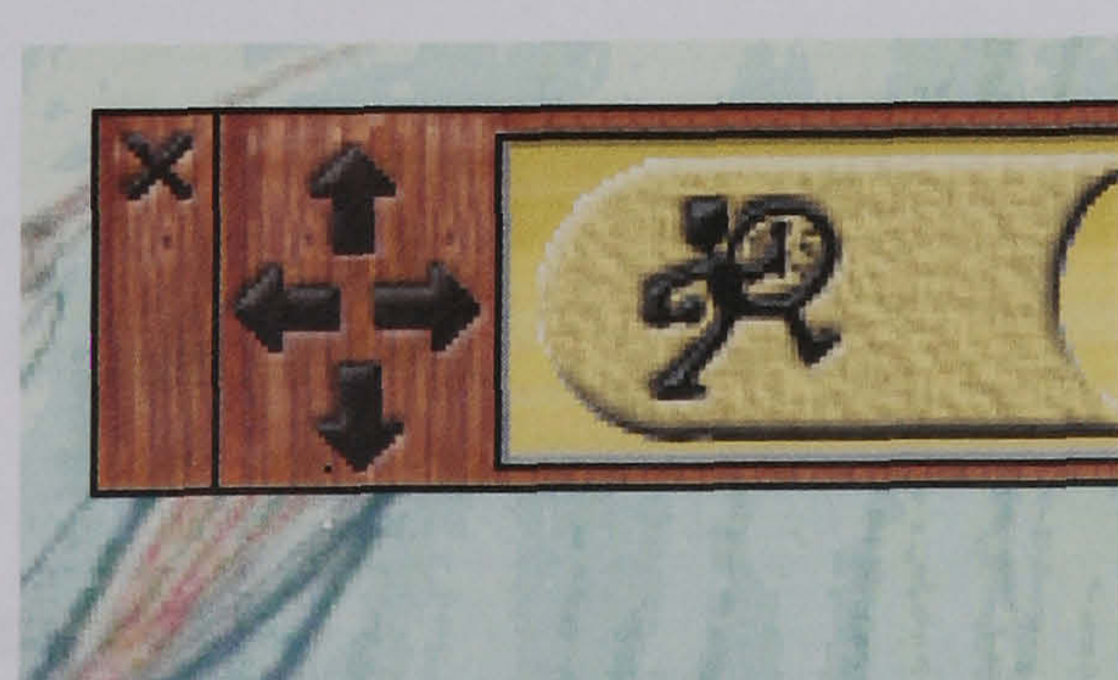


Figure A1.5: The sound stone

Describe what you have done.

A3 The protocol of the task-based interview of the learning investigation

Step 1

Tasks

1. Switch on the game.
2. Imagine:
    a. What are the rules of the space kid?
    b. What are the rules that make the space kid work?
    c. What makes the scorer go up and down?
3. Switch off the game
4. Find out: a. What are the rules of the space kid, the red ball, the blue ball, the bouncing ball? b. What do scorers count?
5. Use the stones to make the space kid to play a sound when goes up and to play a different sound when it goes down.
6. Switch on the game again.
7. Describe what happens.
8. Can you make changes in order the space kid will move easier up and down?
9. Demo:
    a. Move the balls not to a position of symmetry. How do you think this effect the result of the game?
    b. Make another ball. See the rules that are the same. How does this affect the result of the game?

Step 2

Tasks

1. Switch on the game
2. Imagine the rules of the two mines.
3. Switch off the game.
4. See the rules of the mines.
5. If you want the space kid not to touch the two mines what will you do? Why?
6. Try it out…What did happen? Why do you think this happened?
7. If it doesn't work, as you want it, what else can you change?
8. Suppose that you want the red score twice as much as the blue scorer / ten times as much. What can you do?

Step 3

Tasks

1. See the rules of the bouncing balls and the bricks.
2. Switch on the game.
3. What will you change in order to the space kid stays nearly the yellow line? Try it out!
4. What did happen? Why do you think this happened? If it doesn't work, as you want it, what else can you change?
5. What will you change now in order to make the space kid reach the blue planet/the red planet? Why?
6. What did happen? Why?
7. If it doesn't work, as you want it, what else can you change?

Step 4

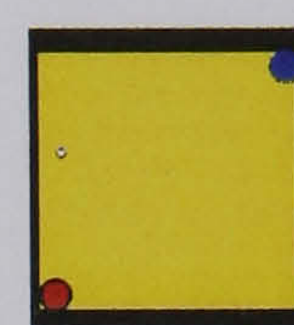
Tasks

1. Having in mind the lottery machine game, can you build a new game?
2. What will you do?
3. What is the role of the lottery machine in the game?

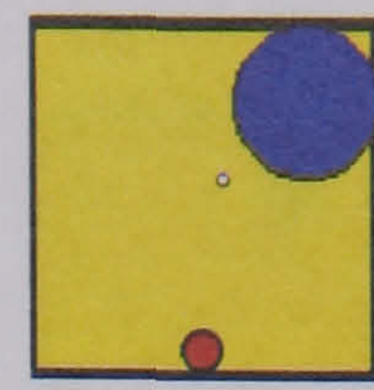
A4 An example of a coded transcript

Line			Codes	Notes/Screenshots
		Step 1		
1	J:	There is a yellow triangle (the space kid) that	A1.1	
2		moves up and down…here (on the red scorer) we		
3		have four points.		
4	R:	When does it (the scorer) get some points?		
5	J:	Is it when it (the white ball) touches the balls?		
6	R:	Maybe…lets see…How does the white ball move?		
7	J:	By itself…Ah, because the reds were less it	D1.1	*Showing the red*
8		touched the red and got a point.		*balls on the screen*
9	R:	What about the space kid?		
10	J:	It went down.		
11	R:	When does it move down?		
12	J:	When we have more blue points.	D3.2.2	
13	R:	Why did it go up now?		
14	J:	Because it (the white ball) touched the red ball.		
15	R:	Can you imagine the rules of the space kid (s.k.)?		
16	J:	When the little white ball touches at the red ball,	B1	
17		our s.k. moves up and when it touches the blue one		
18		it moves down. The red points are more.		
19	R:	Can you find the rules and read them?		
20	J:	When the s.k. receives a blue envelope it moves	B1	
21		down and when it receives a red envelope it moves up!		
22	R:	Ok! Who do you think is sending these envelopes?		
23	J:	The scorers…		
24		*She opens the rules.*		
25	J:	When you start the game you have zero points, when it gets a red envelope it gets one point.	B1	
26	R:	What about the blue scorer?		
27	J:	When it gets a blue envelope it gets one point, as well.		
28	R:	Where these envelopes are coming from?		
29	J:	May be from these balls here.		
30		*She opens the rules.*		
31	J:	When it touches something, it bounces it and gives a blue envelope.	A1	
32	R:	What about the red balls?		
33	J:	It's the same rule; we just have a red envelope here.		
34	R:	What else do we have in our game to look at its rules?		
35	J:	This little white ball here…		
36		*She opens its rules.*	B2	
37	J:	There are no rules…		
38	R:	Yes, it only has a speed that you can find it by		
39		using the star It only moves in the yellow square		

	Ok…Let's start the game…	
J:	Because the white ball touched the red ball our s.k. moved up and because it touched the blue it moved down.	A2
R:	Can we make use of the stones so that the s.k. makes a sound when it goes up and a different sound when it goes down?	
J:	Yes…	
	She puts the stones and switches on the game…	
J:	When it touches the blue ball, [the s.k] goes down and makes the noise that I chose.	B1
R:	What are the points now?	
J:	Two-Three…Four-Four. They are the same.	
R:	What happens to our s.k. when we have the same score?	
J:	It's on the yellow line, at the place that it was in the beginning.	A2
R:	Ok…now I will switch off the game and I will put this blue ball here and that red ball…	
J:	Put it here!	
R:	Where?	
J:	Here (opposite the other ball).	D3.1.1

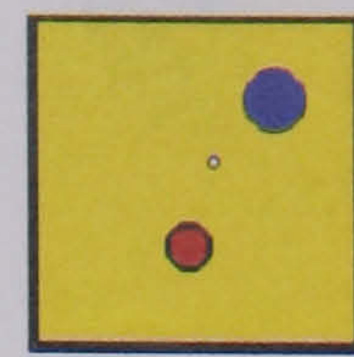


R:	What will it happen if I put it there?	
J:	The same thing as before, but the balls will be in a different place.	D3.1.1
	She switches the game on.	
R:	What happens?	
J:	It is in the middle, down, down…	
R:	Which ball will it (the white ball) touch now?	
J:	The red one…We have 3-3.	
R:	It's in the middle now….	
J:	It went a little down and it has five for the blue and four for the red scorer.	
R:	I will stop the game now and I will put the red ball here. What will happen?	
J:	The same, but the balls will be in different places.	D3.1.3
R:	Where do you think our s.k. will move?	
J:	Up and down.	
R:	Will it be near or far away from the yellow line after one minute?	
J:	It will be in the middle, may be a little bit up or little bit down. Depends on this white ball.	D.3.1.3 D1
R:	How does this come?	
J:	It moves by itself. It goes in different places and if it goes down and there is a red ball there it touches it and our s.k. moves up.	D1.1
R:	Do you know where the white ball will move?	
J:	No…I just know that when it is very near to the red ball, it might touch the red ball.	D1.1

	She gets the star and makes the blue ball bigger than the red one.	D2.3 D3.2.2	
J:	It will move very down because the blue is bigger and it (the white ball) will touch most of the time on it, because the ball takes more space in the yellow square.		
R:	More space?		
J:	It's bigger and takes more space, and before it (the white ball) went there and didn't touch the ball, now it will go there and it will find and touch the ball.	D3.2.2	*Spatial quantity & probability*
R:	Can you show me how the white ball moves?		
J:	I will first need to start the game.	C	
	She switches the game on.		
J:	The blue wins.		
R:	If I want my s.k. to move faster up and down, what can I do with the balls?		
J:	I can make the red bigger as well.	D6	
R:	And then?		
J:	The scorers will be equal.	D3.1	
	She takes the star and she makes the red ball bigger.		
J:	I think the red will win.		
R:	Why is that?		
J:	I think I made it a little bigger than the other… We can open the game and if the scorers are the same that means that they have the same size, otherwise the one is bigger than the other.	D3.2	
R:	What about the s.k.?		
J:	If it is as now that means our balls have the same size…		
	She starts the game.		
J:	You see, now it means that they have the same size. Our space kid is near the yellow line.	A2 D3.1	
R:	How long do we need to leave the game on?		
J:	Four seconds…may be…I don't know. We need to see. But now it moves quickly. The white ball goes many times on the circles (the big balls), because they are big. Now the blue has more points, now the red….the blue…the red.	D6 D6	
R:	For how long do we need to leave it going?		
J:	We need more seconds…The red now is 91 & 80…	D6	
R:	If we change the position of the balls, will the result change?		
J:	Yes…I think so…may be…	D3	
R:	Will it be the same?		
J:	I don't know. May be.		
R:	Can you make an arrangement in order to have the same result?		

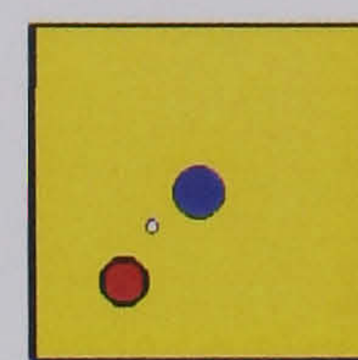
J: Ok…let's see…

She changes the size of the balls. The blue is bigger than the red ball. D3.1.2



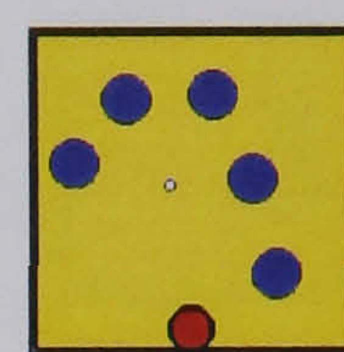
J: We might get the same numbers with this arrangement, but it's more possible for the blue to win because it's bigger. Let's see…shall we start the game? Let's start it! It might get more on the red. The white ball is near the red one, but at the end the blue will win. It's bigger…I need to make them to have the same size in order to get equal points. D3.2.2

She makes the blue smaller than before. She starts the game.



R: Ok! But, I think it moves too slowly. Can you tell me what it will happen if I copy more blue balls?

I copy some blue balls.



J: The blue will win. The blue has more balls, four more. D3.2.2

J: We have 5 blue and 1 red balls.

R: So, if the blue gets 5 points, how many points will the red have?

J: It will have one point. D5

R: And if the blue gets 10 points?

J: 5? May be…because the blue got 10 then the red will get 5. D5

R: Where will the s.k. go?

J: It will go down. The blue balls are more, so it (the white ball) will touch on the blues and it (the space kid) will move down. Let's see… D3.2

R: Oh…it moves down. How are you going to characterise the white ball?

J: It moves quickly… D6

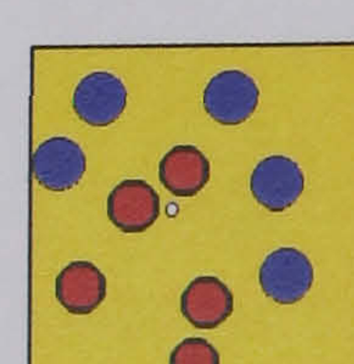
R: Can you do again something with the balls in order for the space kid to move near the yellow line?

J: Yes…I'll copy another 4 red balls. So, we're having 5-5. Let's see… D3.1

R: How did you arrange the balls?

J: I put three red balls near each other and two far away and two blues, two blues and one alone. Oh…it goes down. D3.1.3

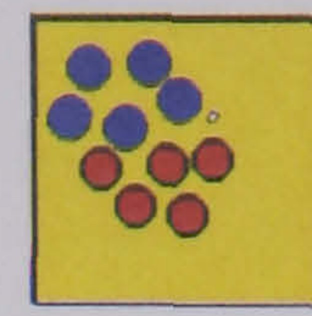
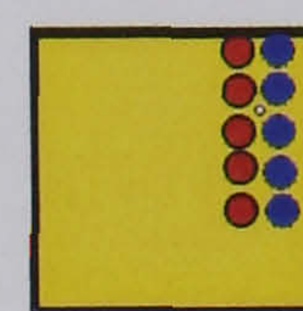
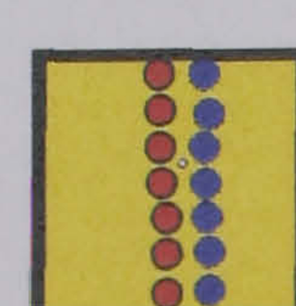
She watches the space kid on the screen.



R: Can you do something to stay near the yellow line?

J: Yes! I'll put near all the red balls and then all the blue ones. Then, I will take these two teams and I will put them near to each other. D3.1

R: That sounds a nice idea!

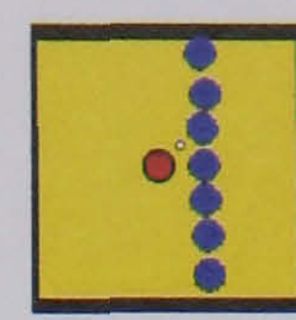
176	J:	Ok! (She makes the arrangement).		
177		*She starts the game.*		
178	J:	Oh! They are equal!		
179	R:	Great! Which colour ball will the white ball touch now?		
180	J:	The red one. The red has 32 points and the blue 27.		
181	R:	But, we want them to be equal.		
182	J:	It's going to get red… here it is… you see!		
183	R:	Can you do something to make them equal (the		
184		scorers)?		
185	J:	Oh…it (the space kid) went up…	A1	
186		*She switches the game off.*		
187	J:	I will put them on a line, one red and one blue. I	D3.1.1	
188		will put the red one on the left and the blue one on	D3.1.1.1	
189		the right…	D3.1.1.2	
190	R:	Why is that?		
191	J:	Because I will put them near to each other in the	D3.1.1	
192		middle. So, when the [white] ball goes to touch		
193		one, it will touch also the other that's near it. So, it		
194		will touch both of them and we will have equal		
195		points. We do not know where it (the white ball)	D1.1	
196		will go, but if it touches one ball it will touch the		
197		other as well.		
198	R:	That sounds a nice idea…can we try it?		
199	J:	Because our ball is small I will put them very near to each other…to have equal points. But, we have to have in mind where we will put the white ball.	D3	
202		*She starts the game.*		
203	J:	Ah! It's getting points very quickly. We have the	D6	
204		blue ones with more points…. now the red ones….	D3.1	
205		Oh! The reds have more points!		
206	R:	But you put them near to each other.		
207	J:	Yes, the red is outside. So, it's touching the white	D3.1	
208		ball more times. I will put them like this now. I		
209		will put the two lines in the middle. Now, I need		
210		another two balls. I'll get the magic wand to get		
211		more balls. It's the good fairy that gave it to us…		
212		*She copies more balls.*		
213	J:	…The little fairy likes us, because we are learning	C	
214		it to play games. Ok…I copied the balls…		
215		…		
216	R:	Do you know how many balls do we have?		
217	J:	Yes…they are equal. I know they are equal, but I	D3.1	
218		don't know how many balls I have.		
219	R:	How do you know that they are equal?		
220	J:	I copied one red and one blue each time. It doesn't	D5.1	
221		matter actually how many they are. They are equal.		
222		*She switches the game on.*		

R: Ah…10-10!

J: Yes…now we have more blue balls…it should go more times to the red balls now…Ah! They are equal!…Now we have more reds…they're going to be equal again! Yes! D4.1

….

R: Let's stop the game now. I will leave one red and seven blues. Can you tell me after a while what number the red will get and what the blue one? D5



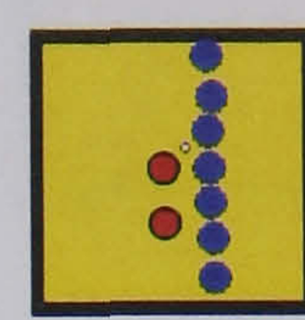
J: How much time will we leave it go?

R: As much as you think…may be one minute.

Explaining intervention

J: Ok! The blue might have 200 and more and the red 100. D5

R: What about if I put another red ball (7B 2R)? If our white ball touches 9 times on the balls, how many times do you think will it touch the reds and how many times the blues?



J: May be 5 blues and 4 reds…or….may be 3 reds and 6 blues…or…may be 7 blues and 2 reds…I don't know. D5

R: Can we also have 7 reds and 3 blues?

J: May be…let's say if you put the balls here, we might have it.

R: Ah…I will put the balls in the position that you used in order to have equal scores. If our white ball now touches 90 times on the balls, what will we get for the red scorer and what for the blue one?

J: May be 60 blues and 30 reds. D5

R: If I have 3 reds and 6 blues?

J: May be 50 blues and 40 reds.

R: If I have 4 reds and 5 blues?

J: May be 20 reds or may be all blues…I don't know. D5

R: Ok…Now, I want the blue scorer to have twice as much as the red scorer. How many balls should I have?

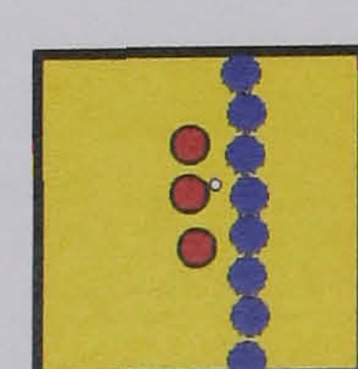
J: I can have 3 reds and 6 blues…So, lets take away one from the red and get another blue… D5.2.1

J: …let's take away 2 blues, but we may get more reds…it's possible.

R: If we want to make it more possible to get twice as many blue balls as red balls? What shall we do?

J: We may have 7 blues and 3 reds.

She copies some balls and switches on the game (8B 3R)



J: We get more blue balls. If we had a bigger white ball it would touch both blue and red balls without passing though… D5

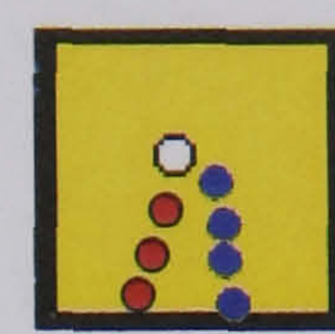
R: We can make it bigger…

J: I mean the white ball! D2

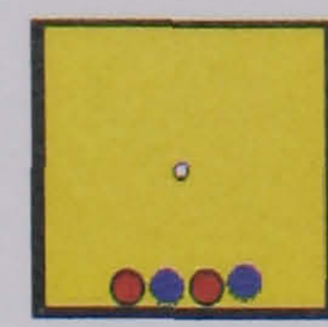
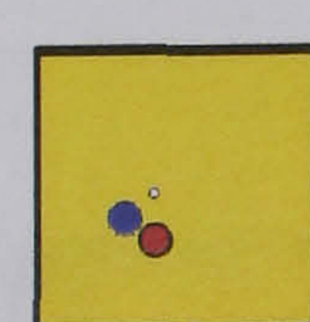
R: Yes! We can use the star!

J: Yes! I remember that window will open and I will

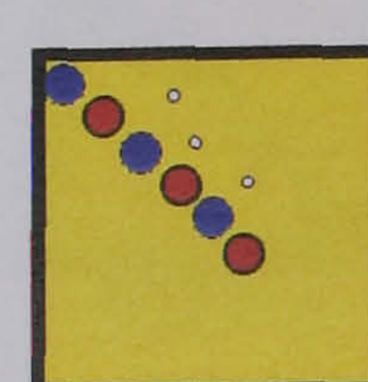
	make it bigger.	
	She uses the star to make the white ball bigger, then, she takes away balls.	D2.2
R:	How many balls do we have now?	
J:	We have 3 more blue balls than red ones. We have 6 blues and 3 reds.	
R:	Why is that?	
J:	We want the blue scorer to be twice as much as the red one…wait a minute.	D5.2
	She re-arranges the balls.	
J:	I will switch on the game now!…Ok! I'll do something else.	
	She makes 4 blues and 3 reds.	



R:	What did you do?	
J:	I'll have more blue balls.	D3.2.2
R:	How many more?	
J:	More than reds…The blue will get more points, that's for sure.	D3
	Step 2	
R:	Let's switch on the game and tell me what's happening.	
J:	We have 5 reds and 4 blues (on the scorers).	
R:	Why did the game shut down?	
J:	It (the space kid) moved up and shut down.	
R:	Did it touch something?	
J:	It touched that red thing (the red planet).	
R:	Start the game again. Can you imagine the rules of the red planet?	
J:	When it has more points the red scorer, the s.k. touches the red planet, it explodes and the game shuts down.	D3.2
	She opens the rules.	
J:	When it touches the s.k., it explodes and the game shuts down.	C
R:	What about the blue planet?	
J:	When it touches the space kid, it explodes and the game shuts down. They are the same as the red ones.	C
R:	Switch on the game again and lets see what happens.	
J:	It (the space kid) touches the red planet.	
R:	Why does it touch the red planet?	
J:	The red balls are more…	D3.2.2
R:	Can we do something in order for the space kid not to touch any planet?	
J:	Yes! We can put the same amount of balls.	D3.1.1
R:	The same amount of balls…where will the s.k. be?	
J:	In the middle.	
R:	Can you make that?	
J:	Yes…or…we can have only one or two more red	D3.1

	balls, but not too many more…we can leave only one red and one blue.		
	She removes some balls.		
J:	I know why it went to the red planet. The red balls were more and it touched them more times…you see…I destroyed so many red balls and there are still more…Ok! I will leave two blue balls and two red ones. I will put one red, one blue, one red, one blue.	D3.1 D3.1.1.2	
R:	Ok…		
	She starts the game.		
R:	How is the white ball moving around?		
J:	It moves up and down, right and left and when it touches one ball and gets a point it then goes everywhere in the yellow square.	D1	
R:	Does it know where it goes?		
J:	It knows.	D1.2	
R:	How does it know?		
J:	It does know, we don't, but it knows.	D1.2	
R:	How does it move?		
J:	For us it moves randomly, but it might know where to go. We don't know where it goes because we didn't make it to go somewhere…Look 11-11!	D1	
	…		
	But, it stopped! Why?		
R:	Do you want to leave it a little more time to see if we have equal numbers?		
J:	Yes!		
R:	Ok…let's move these two planets. You are going to move them as far as you want to.		Experimental intervention
J:	Ok! I'll put them here (further away from the space kid, but not at the edge of the screen).	D6.1	
	She starts the game.		
R:	It moves too slowly.		
J:	I will put some more balls.		
	She copies some balls.		
R:	How many balls do we have now?		
J:	We have 4 blues and 4 reds.	D5.1.2	*4:4, 1:1*
R:	Let's see how you are going to arrange them.		
J:	Ah! I know!		
	She destroys some balls, leaving 1 red and 1 blue.	D5.1.2	
	…		
J:	I will put them like this…Because the white ball is in the middle, it touches the red and blue ball. Have you imagined filling in our square with blue balls and having one red in the middle?	D3.2.1	
	She laughs…		
R:	What will happen?		
J:	It will kill itself all the time on the blue planet!	D3.2.1	
R:	Will the red get a point?		
J:	It will, but the blue will take much more.		

369	R:	We can try that afterwards…Look! It keeps	
370		moving slowly. What shall we do?	
371	J:	Add some balls?	
372	R:	If you think so…	
373	J:	Look! 20-20!…Ok! I'll put some balls…Like	
374		this…a diagonal line.	
375	R:	Oh…you copied another white ball (accidentally),	
376		what do you think will happen?	
377	J:	I don't know. We will get equal points.	
378	R:	Will we get points more easily or more difficult	
		than before?	
379	J:	We will get easier points, because these two balls	D2.2.2
380		are moving together, collecting more points…Can	
381		we destroy these? (the two planets).	
382	R:	Yes we can. Do you want to?	
383	J:	Yes…they shut down the game and it doesn't stay	D6
384		for a long time.	
385	R:	Can we make it stay longer without destroying	
386		them?	
387	J:	Yes…if you put more white balls, what will	D6.1
388		happen?	
389	R:	Can you tell me?	
390	J:	I will put some more…we will have many points,	D6.1
391		changing all the time.	
392	R:	What about our space kid?	
393	J:	It will keep going up and down, up and down and	A1
394		it will get a headache!	D6
395		*She copies some white balls.*	
396	J:	At the end it will kill itself…it will have a	
397		headache all the time!	
398	R:	Where will it go to kill itself? At the red or at the	
399		blue planet?	
400	J:	At both of them! We won't know where to go!	D3.1
401		*She keeps adding and destroying balls.*	
402	J:	We do not want these two balls…	
403		*She put one red, one blue etc, and they remained*	
404		*two blues…*	
405	J:	Wait a minute!	
406	R:	What's up?	
407	J:	I'm checking whether the balls are equal.	D3.1 D5.1.2
408	R:	Are they?	
409	J:	Yes!	



410		*She switches on the game.*	
411	J:	Oh! You cannot understand from the	A1
412		numbers…they keep moving too quickly. We	
413		should look at the space kid…Oh! It went up…92-	
414		110. We should move the planets again.	
415		*She moves the two planets to the edge of the*	D6

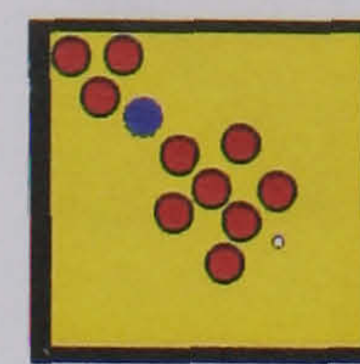
screen.

R: Let's start the game again.

J: Oh! I think it will go on the red. Yes! It's on the yellow line now! They passed the 100…Oh again on the blue…oh 118-137. D4.1

R: That was a very good idea…If we want now to get quickly at the red planet without getting any blue points, what shall we do?

J: I will put the reds here and the blue here…It will only get one point (the blue scorer). D3.2 D7



She starts the game.

J: You see…it got only one point. D7

R: If we don't want to get any blue points what shall we do?

J: We should leave only red balls! D3.2.1

R: Ok! But if we have to have some blue balls?

J: We can have many reds and only one blue. Do you want me to do that?

R: If you would like…

She copies and destroys balls.

J: Oh! It became a drawing. You see…here is his nose, his eyes, his hands…it doesn't have any legs. I'll do him his legs… C2

R: So, how many points you think the blue will have?

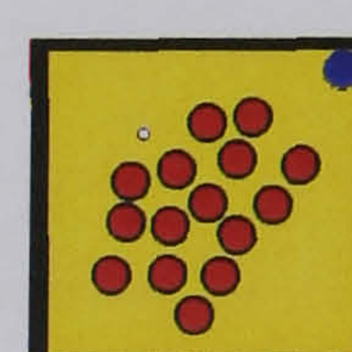
J: Too little…may be the red 100 and the blue, maybe, 20. D5

She switches the game on.

J: The red got 20 and the blue one point.

R: Can you make it in a way to have a blue ball, but not to have any blue points?

J: Yes…I will put the red balls near to each other and I will put the blue one down on the side. I will copy some red balls to make the red to get more space. D3.2 D3.1.3



She starts the game.

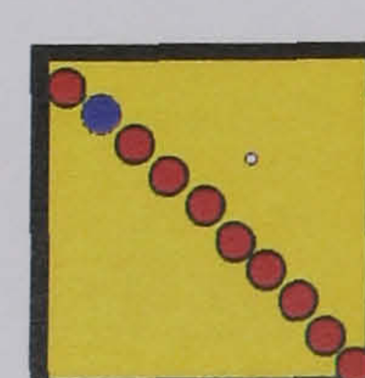
J: Oh! It took one point…I don't know…

R: Do you remember now your diagonal line.

J: Yes!

R: How many balls do we have here?

J: One blue and nine reds.



R: Ok. We have ten points now. How many blues you think we have and how many reds?

J: 10 reds and may be no blue points…or may be 9 reds and 1 blue point. D5

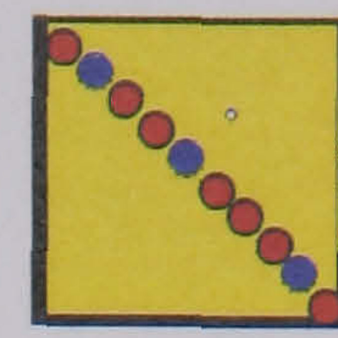
R: If we get 20 points?

J: May be 10 blues and 10 reds.

She starts the game.

J: We got 24 reds and 4 blues. We got more red balls.

R: Ok…now we will have 3 blues and 7 reds. What points will we get?



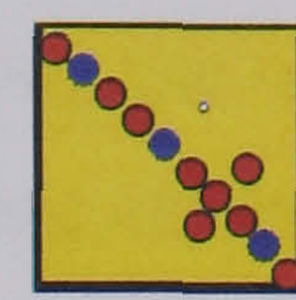
J: We will have…7 red points and 3 blue points. D5

R: If we get 20 points?

J: 20 points? We may have 15 reds and 5 blues.

R: If we need the red to be twice as much as the blue?

J: To put another two reds…eh…



She starts the game…

J: 33-13. I have 9 reds and 3 blue balls.

R: If we want this (the red score) to be twice as much?

J: Oh…I don't know…I can't do it…

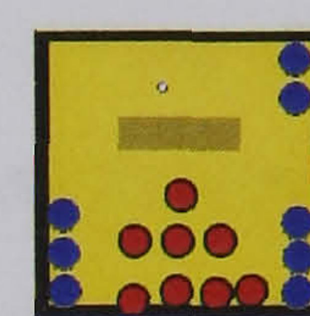
Step 3

R: What happens with the brick?

J: It touches them and moves them away. It has the same rules with the balls, but now messages.

R: Can you do something here in order for the space kid to be near the yellow line?

J: I will take away some red balls. D5.1.3



R: Ok! How many balls do you have now?

J: 8 blues and 8 reds…Shall I start the game? I think it's ok!

Discussion

R: What did you do with these games?

J: We played with a space kid and some balls, and we tried to put as many balls so that as the time passes our numbers become equal. Our space kid should remain in the middle, near to the yellow line.

R: How did the white ball move?

J: It moved…the computer controlled it. We tried to think where might the ball go. D1.2

R: What did we do in order to have a clue about which ball will touch?

J: We put…when we had more red balls, we knew that it would go to the red balls. D3.2

R: If we wanted the space kid to remain on the yellow line?

J: We put the balls on a diagonal line.

R: What did you like from these games?

J: Each thing had its rules and in order to control the white ball we should delete all the rules from the computer that control it, and put our rules on it! B D1 D2.2

R: And if we couldn't change its rules?

J: We put balls…

R: OK, thanks a lot!

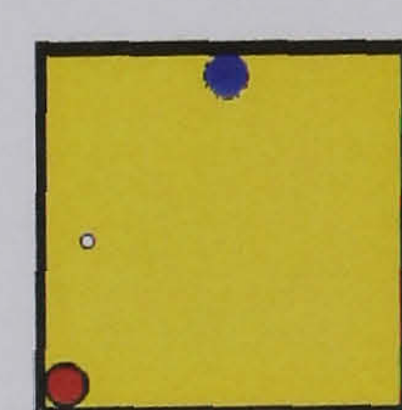
A5 A summary of a transcript

Irene's first reaction to the game was sample space-oriented. She was very curious about how the white ball was moving and at the beginning she expressed many times the idea of controlling this ball, which as she said 'it moves everywhere'. After watching the white ball moving around and concluding that 'It is just moving around without doing anything', she decided to deal with the coloured balls inside the yellow square. The movement of the white ball seemed to be arbitrary and the absence of a rule beside it suggested a movement with no purpose. She realised that she cannot do anything about that, unless she creates a rule for the ball. The arbitrary movement here is an expression of randomness. Her recognition of this led her to begin thinking of controlling the placing the coloured balls, as a way to control this 'free' situation.

Generally, in her constructions she used outcome-oriented description based on the global events on the screen, to value whether her idea was working ok. Because of the continuous movement of the white ball, the change on the scorers came without stopping. When the scorers changed more quickly, looking at the global events in the game was more helpful for her to make decisions, and to express the possibility of something to happen by describing the movement of the space kid, up or down.

Fairness was Irene's first reaction for dealing with two colours. On my intervention about changing the place of the balls, she put the balls in symmetrical positions.

R: .... Let me ask you something. If I move the blue ball a little bit and place the red ball here, what do you think will happen?



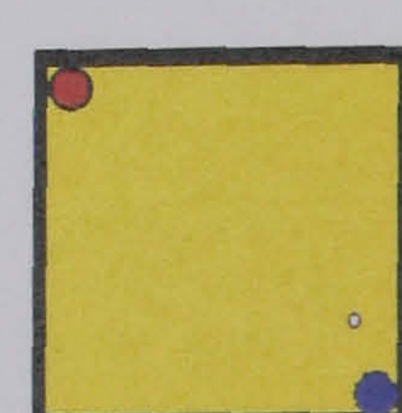
I: It (the white ball) will move again. We just changed the place of the balls.

R: What will happen to the counters?

I: I will do something...

R: Can you please answer my question first?

I: Never mind... I will put this here and that one I will put it here.



R: Why did you put it like this?

I: Because if this gets that one then it will go to the blue and then get the red and go like this and get that.

R: Where will the space kid move?

I: On the yellow line…

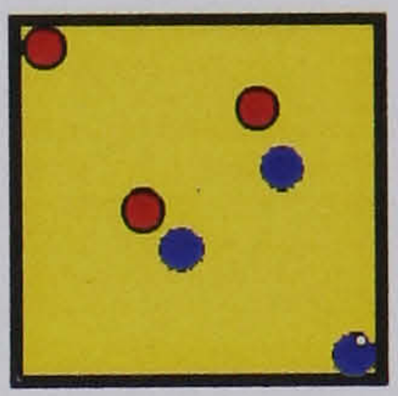
She starts the game.

I: You see! (*She laughs*). We have 0-0 points for the time! 2 points, 3…

Irene ignored my question about the 'unsymmetrical position' of the balls. The idea of fairness comes intuitively, since 'fair' is the thing that 'must' happen in a game. Furthermore, in Irene's mind fairness is connected with symmetry.

During this construction Irene was watching the scorers changing points very slowly. This event made her copy more balls into the sample space. She wanted the results to change quickly, so she copied another two balls of each colour. Perhaps this is also a sign of situated abstraction for the law of large numbers and probability. The placement of these extra balls was again symmetrical: the sample space should be fair, so the balls should be placed in symmetry. Her explanation of symmetrical placement of the balls was as follows:

I: I will put them like this. The one near the other.



R: Why?

I: Because the white ball can get more easily the red and the blue balls and they will get more points.

R: Will they get the same points?

I: No! Because the blue has four points and I placed the ball here from the beginning, so it will get more points.

R: Can you make something to get the same points?

I: Yes…in the middle. (She places the white ball in the middle).

R: Will they get the same points now?

I: I don't know. Yes! I think so, may be. Shall I start the game?

R: Yes!

She starts the game.

I: 18, 19…hey move! Wow! They got the same points. Yes! 27! (*She is screaming*). 29-29 Wow! 33-33! It's so quick!

Apart from thinking about the symmetrical position it seems also that the starting point of the white ball has a role in Irene's mind, at least at the beginning. She tried to guess the first movement of the white ball to make a decision about how the ball would move and which balls it would touch more frequently. Her decision worked for the short-term movement of the ball but not in the long term. Thus, the continuous movement of the white ball gives two levels for movement: the short term, which might be expressed as 'local', and the long term movement, which might be expressed as 'global'.

Irene's enthusiasm was evidenced by her continuous shouts while she was watching the scorers on the screen. At this point I asked her to place more balls in the sample space. This is when she thought of the idea of surrounding the white ball, 'I will surround this white ball'. She did this again by placing an equal number of blue and red balls, five of each colour. After some trial and error with spatial arrangement she concluded by constructing a semi-circle, placing the blue and red balls in an alternating pattern. The method of trial and error, in a spatial representation environment, allowed her to develop the idea of building a fair sample space; this requires not only the same number of balls of each colour, something that she knew from before, but also a particular spatial arrangement.

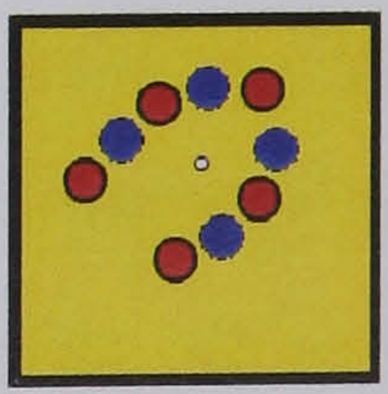
I: I will get this blue ball, put it where the red was and I will place the red near to it, such as to have blue, red, blue…

R: Why are you doing this?

I: I am making it like pattern because with this way they could have equal points! Is it right?

R: There is no right and wrong… I just need your ideas!

I: Ok… I think now it's ok!

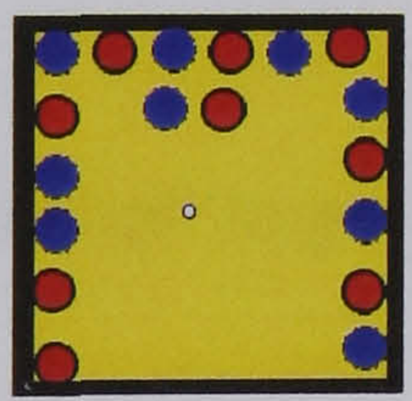


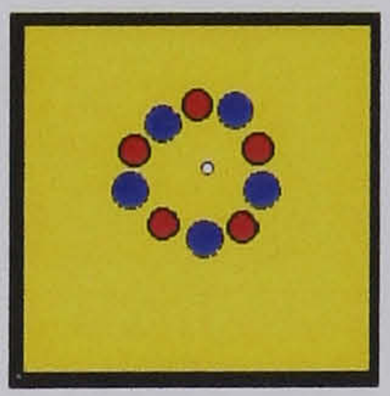
She starts the game.

I: Oh… you bad red balls! Oh…look! 21-21! Something happens! Look we have equal numbers! (She laughs). They got 61 points!…Oh…it is ok! Wow…102! 104! Oh it moved down… 112-122… Oh, the poor space kid! It moves up now! Oops! A… It seems to move up now… Ahhhhhhhhhh 207-207! (*She laughs!)*

The starting place of the white ball still played a role in Irene's thinking. She finally came to the idea of having the white ball in the middle and surrounded by the other balls. Although the starting place of the ball didn't play a significant role in this construction, she evidently felt it to be important for fairness.

In the previous constructions and in this one, Irene faced some unfair situations. She started by making a circle of balls and because that was small she had to place some red balls outside the circle. This made her sample space not to work effectively, and the blues to get a bigger score. This is another point where she seemed to find significance in the spatial representation of the balls in sample space. So, although at the beginning she was very careful of having equal numbers of balls inside the sample space, which is how she understood fairness, she made the decision to take away one blue, leaving four blues and five reds and tried it out. The environment made her to question her idea that fairness only means equality between the numbers of the balls of the two colours; she came to coordinate both the spatial arrangement and numbers of the balls to achieve a desired fair result. After another trial of placing the balls in the space she thought of the semi-circle pattern. It seem that at this point she made the connection between the two pieces of knowledge - equal numbers and the placement of the balls inside the sample space. After this (in phase 2, where she tried to make the space kid not to touch the two planets) she constructed another two fair sample spaces as follows:





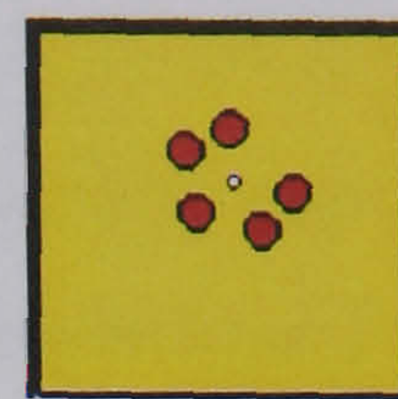
An effective strategy devised by Irene for constructing a fair random environment came to be the 'surrounding' of the white ball and making a pattern with equal numbers of balls of different colour. This was her way to 'control' randomness and to make the white ball to 'touch both blue and red balls the same times'.

The construction of a fair environment was not made by Irene because she was asked to, but because as she said is 'better for both colours to get the same points on the counters'. That statement was made when she had tried out different representations and had got an unfair sample space. The idea of fairness is something that Irene developed through her experience in the games. As Irene was getting bigger scores on the counters, she seemed to judge differently the equality of fairness. As the above snapshot shows, while the scorers were changing very quickly, she expressed that 102 and 104 are equal, and then 112-122 also declared equal.

Irene easily constructed an 'impossible' sample space, where no blues would give any points for the blue counter.

R: Can you do something in order our space kid to move quickly to the red planet?

I: Yes… Now, it won't get any blues.



R: Are you sure about it?

I: Yes!

She starts the game.

I: It will explode on the red… They are alone…so that's ok!

R: If we need to have some blue balls as well?

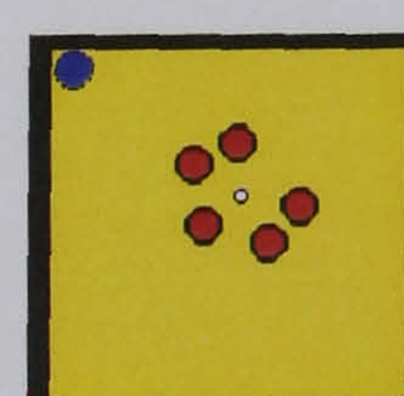
I: I will put them somewhere that the ball doesn't go.

R: Is there such a place?

I: No, but I can make one. We could also put only blue balls and move to the blue planet. Anyway, I will put only one blue ball.

R: Why only one?

I: To be easier for the ball to get the reds.



She starts the game.

I: It might get some points, but it will make an explosion.

Notice here that although Irene had the idea of making it difficult to get any blue points, she understood that it is still possible for the white ball to touch the blue ball. There is an implication here that perhaps Irene thinks that everything is possible to happen. Even extreme variability is a possibility. This is also a point where she sees this construction in the long term, and reveals thinking about the 'law of large numbers'. This was also found in the previous episode where she constructed a symmetrical fair sample space of three balls of each colour. When she was asked whether the two scorers will get the same points, she said 'I don't know. Yes! I think so, may be. Shall I start the game?' Although here she seemed to be almost sure of the result, the reason that she could not control the movement of the white ball made her think about the possibility of things to happen (or not happen) which she did (not) want to happen.

Irene worked with patterns to place equal numbers of balls inside the sample space. She did not care about the number of balls, only to have the same number of each colour. She did not count the balls to see if they are equal, but she used patterns to produce the same number of balls. This may indicate that she had the knowledge of 'equality of an event', such that 1:1 is equal to 5:5. Most of the times she preferred adding balls in the sample space instead of taking balls away. The total number of the balls (the denominator for the probability) did not play a role in her construction of fairness, only the equality of the numerators.

When she had to deal with big numbers and to express ratios requiring double figures, she tended to express the outcome by saying only which colour is going to get more or less points.

R: Ok... So, they are 10 balls, 9 blues and 1 red. If the red gets 10 how many will the blue get?

I: 19... Because the red is one and the blues are 9 and they surround the white ball.

R: If now the white ball touches 100 times on the balls, how many times will it touch on the red and how many on the blues?

I: On the blue it might touch 88 times or 100 and on the red 78...

Irene here expressed large and small numbers, depending on which colour it was possible to win, without even thinking whether the total of these numbers is the total number of hits by the white ball. She used the words 'more' and 'less', and does not appear to think in a proportional way to express a possible number result.

Irene made some effort to understand how the game was working. She used most of the time the third person to express the rules of objects. She did not express the rules as 'there is an object having this rule', but she did express about what an object was doing in the game. She was self-oriented when she was describing rules. For example, when she described the rules of the counter she said, "See. The first rule is 'when I start the game it's zero and when it moves down and gets the red message it adds one.'" Although that she did not express the rules of the game using the third person she understood very quickly how the game worked and connected easily the local events that happened inside the yellow square with the outcome they had for the space kid. The use of the third person seemed to be only a problem of expression, that might have something to do with the age of the children, and not a problem of understanding the game.

During the game, apart from changing the balls inside the yellow square Irene also had the idea of changing the size of the planets and moved them far away from the yellow line in order for the space kid not to touch them. This was done to give time for the space kid to make more movements and to remain near the yellow line. This suggests an intuition by the child that is connected with the law of large numbers.

A6 A sketch of a transcript

A. Local and Global Thinking

DESCRIPTION	PAGE ON THE TRANSCRIPT (the line was highlighted on it)
A1. 1st description s.s. oriented+outcome oriented (mix)	Pg1
- Connection between local and global thinking	Pg 1
A1.1 Global oriented	Pg 2
A1.1Need the use of global thinking	Pg 3
A2 Global thinking of equality	Pg 4
A2 Global thinking of proportional thinking	Pg 5

B. Expressing the rules

3rd person	Pg1
1st person	Pg 6

C. Engagement with the game

C. To have fun	Pg 9

C. Expressions of random mixture

D1. Unsystematic movement	
D1.*: Liberal movement	Pg 2, Pg 6, Pg 8,
D1.*[32]: The starting point of the white ball plays a role	Pg 2, Pg 9
D1*: Moving in a strange way	Pg 10

D2 Changing the mechanism	
D2.4 Have another colour of balls	Pg 2
D2.4Make a rule to the scorers to get 50 points	Pg 3
D2.2.1 More white balls	Pg 3
D2.2.3 Changing the speed of the white balls	Pg 4
D2.3Giving speed to red and blue balls	Pg 4
D2.4Making new rules	Pg 7
D2*Having only the white ball	Pg 7

D3.1 Construction of fairness	
D3.1.1 Moving and changing the number of the elements of sample space	Pg 2
D3.1.*Same number and size of balls	Pg 3
D3.1.1.1 Symmetrical teams	Pg 4. Pg 5
D3.1.2.* Mixture	Pg 4, Pg7
D3.1.2.* Same size	Pg 8
D3.1.*Spatial representation/distribution	Pg 5

32 *: A not pre-defined code on the coding system (see Chapter 4, section 4.3.5).

D3.2 Construction of unfairness	
D3.2.*The same size of all the balls/different number	Pg 3
D3.2.*Distribution: 'it's how I put them'	Pg 5
D3.2.*'No points=no red/no blue balls	Pg 9
D3.2.1Certain and Impossible events/no red balls	Pg 9

D4 Judgement of equality in fairness	
D4.*'Someone might get more, but they will be equal…'	Pg 2, Pg 3
D4.*A small difference 40-43	Pg 3
D4.*Almost the same	Pg 5. Pg 7
D4.*Trial and error	Pg 2

D5 Proportional thinking	
D5.*'The more will win…'	Pg 5
D5.1.2 '7/1=70/10 only with one ball…'	Pg 5
D5.2 'Twice as much, 2-1' – only with small numbers	Pg 5
D5.1.2 ¼=10/40, 1/5=10/50	Pg 5
D5.*Taking away balls to make them equal	Pg 6
D5.*'To be sure for his decision'	Pg 6

D6 Infinity	
D6.1.*More balls	Pg 3
D6.1.*Move planets at the edges	Pg 6/ Pg 7
D6.1.*Change the speed	Pg 8
D6.1.*Adding balls of both colours	Pg 3

D7 Possibility	
D7.1.* 1 red ball	Pg 9
D7.1.*Only blue balls	Pg 9